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SYNTHESIS OF ADMITTANCE MATRICES USING RC NETWORKS AND OPERATIONAL AMPLIFIERS

A THESIS

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SUMMARY

This thesis is concerned with an investigation of the use of operational amplifiers as the active elements in active RC synthesis procedures. Resistors, capacitors, and operational amplifiers are used as network elements for the realization of N x N short-circuit admittance matrices and simultaneous realization of N short-circuit admittance parameters. The results of the investigation can be summarized in the following four theorems:

Theorem 1

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless active RC network containing ideal operational amplifiers (a) it is sufficient that the network contains N ideal operational amplifiers; and (b) if the matrix possesses a kth order pole of rank N* off the negative-real axis, it is necessary that the network contains N ideal operational amplifiers.

Theorem 2

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless grounded active RC network containing ideal operational amplifiers, it is sufficient that the active network contains 2N ideal

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*By "a kth order pole of rank N" it is meant that the matrix of the kth coefficients of the principal parts of the Laurent expansion of $\bar{Y}$ about this kth order pole has rank N.

operational amplifiers.

Theorem 3

To realize two arbitrary short-circuit admittance parameters, which are real rational functions in the complex-frequency variable, with a 2-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the network contains one ideal operational amplifier.

Theorem 4

To realize N arbitrary short-circuit admittance parameters with an N-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the RC network contains one ideal operational amplifier provided

(a) the admittances are real rational functions in the complex-frequency variable and

(b) one admittance parameter is prescribed from each column (row) of the short-circuit admittance matrix of the active N-port network.

Each of these theorems has been proved, and experimental verification of the realization procedures has been accomplished by using commercially available operational amplifiers to approximate the ideal ones utilized in the synthesis procedures.

In order to prove Theorem 1, the N x N admittance matrix of an N-port transformerless active RC network with m ideal operational amplifiers embedded in it is expressed in terms of the parameters of the (N+2m)-port passive network. It is then shown through the use of an argument on the rank of the matrices that m must be greater than or equal to N if the active network is to realize certain classes of admittance matrices. The realization procedure is accomplished by equating the prescribed admittance matrix to that derived from the network with m = N and identifying the parameters of the passive network so as to satisfy the equation. Also, the parameters must be identified so as to result in a realizable short-circuit admittance matrix. Two realization procedures, both of which yield balanced networks, are given for realizing the N x N short-circuit admittance matrix.

Proof of Theorem 2 is accomplished by letting m equal 2N and equating the prescribed N x N short-circuit admittance matrix to the admittance matrix derived from the network. The parameters of the passive network are then identified so that the resulting matrix can be realized as a transformerless passive (5N+1)-terminal RC network of two-terminal impedances with common reference node and no internal nodes.

Theorem 3 is proved by developing six realization procedures--one for each of the six possible pairs of prescribed parameters. These procedures are then generalized and used to develop a realization procedure used to prove Theorem 4.

Numerical examples are included to illustrate the three realization procedures developed in the proofs of Theorems 1 and 2 for the case N=2.

CHAPTER I

INTRODUCTION

In the past decade considerable interest has been expressed in the synthesis of networks to have prescribed short-circuit admittance matrices. Because of the size, weight, and expense of magnetic elements, it is desirable to avoid their use in these networks for practical reasons. Also, the inductance and ideal transformer have proved to be unsatisfactory elements as far as their approximation to their respective mathematical models is concerned. The rapid development of the transistor has stimulated considerable interest in active network theory, and low-cost active elements have equipped the synthesist with the means of replacing the inductor and transformer and, in addition, to extend the range of realizable functions far beyond that possible with passive RLC networks.

Active network refers to a network that is lumped, linear, and finite, but not passive and not necessarily bilateral. Among the devices which have been employed in active RC synthesis are the gyrator, controlled source, negative impedance converter, and negative impedance inverter. All of these devices possess at least one feature in common--they can all be realized with resistors, capacitors, and one or more operational amplifiers. Thus, it would seem more efficient to utilize the operational amplifier as the active device in active RC synthesis.

Kinariwala[4] has shown that any driving-point function can be realized with one ideal amplifier, whose gain is determined by the synthesis procedure, embedded in a passive RC network. When the gain of the ideal amplifier is specified in this manner, he is essentially dealing with a voltage-controlled voltage source. The RC controlled source synthesis problem was later solved when Sandberg[1] proved that an arbitrary N x N matrix of real rational functions in the complex-frequency variable (a) can be realized as the short-circuit admittance matrix of a transformerless active RC N-port network containing N real-coefficient controlled sources, and (b) cannot, in general, be realized as the short-circuit admittance matrix of an active RC N-port network containing less than N controlled sources. Furthermore, he has shown that an arbitrary N x N matrix of real rational functions in the complex-frequency variable can be realized as an unbalanced transformerless active RC network requiring no more than N controlled sources.[2] The passive RC network required in this realization can always be realized as a (3N+1)-terminal network of two-terminal impedances with a common reference node and no internal nodes. Sandberg's work, particularly his matrix factorization procedure, is an integral part of several of the realization procedures developed in this investigation.

The problem of simulation of transfer functions using an operational amplifier as the active element has received considerable attention. References (5) through (9) all use the same approach to the problem--a network is assumed *a priori* and formulas are developed for the network element values to simulate various transfer functions.

These schemes offer several drawbacks, the most serious of which is the fact that they are all of the trial-and-error type and not of systematic synthesis. Pande and Shulka[11] went somewhat further and developed a procedure for synthesizing a large variety of transfer functions of any order.

The purpose of this investigation is the development of synthesis procedures for short-circuit admittance matrices using only resistors, capacitors, and ideal operational amplifiers as network elements. In addition to the matrix realization, procedures will be developed for synthesizing N-port RC networks which contain one ideal operational amplifier and realize N short-circuit admittance parameters simultaneously. Experimental verification of the realization procedures is also included in the investigation.

Some of the terminology and notation that will be employed throughout this thesis are as follows:

1. A rectangular matrix will be denoted by [A] or $[A_{ij}]$ in which A_{ij} denotes the i,j element (the element that appears in the ith row and the jth column) of [A].

2. A column matrix will be denoted by B].

3. The determinant of a square matrix [A] will be denoted by det[A].

4. A bar over a symbol denotes a submatrix of a partitioned matrix.

5. The degree of the polynomial g is denoted by deg g.

6. The rank of the matrix [A] is denoted by rank [A].

7. The maximum degree of the polynomials which are elements of [A] is denoted by deg[A].

8. The adjoint of [A] is denoted by adj[A].

9. The transpose of [A] is denoted by $[A]^t$.

10. Lower-case y's are used for the short-circuit admittance parameters of the passive networks, and capital y's are used for the parameters of the active networks.

Additional notation will be introduced and defined as it is needed.

CHAPTER II

N x N SHORT-CIRCUIT ADMITTANCE MATRIX SYNTHESIS USING A BALANCED NETWORK

In this chapter necessary and sufficient conditions for the realization of a short-circuit admittance matrix with a balanced transformerless active RC network containing ideal operational amplifiers will be considered. A network of the type under consideration is shown in Figure 1. First, an equation will be derived for the short-circuit admittance matrix of the network. From this equation, a necessary condition will then be deduced for a particular class of matrices. Finally, sufficient conditions will be hypothesized, and two realization procedures will be given as proof.

Analysis of the Network

Consider Figure 1 and let the following set of definitions be made:

$$\bar{E}_a = \begin{bmatrix} E_1 \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ E_N \end{bmatrix} \qquad \bar{I}_a = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_N \end{bmatrix} \tag{1}$$

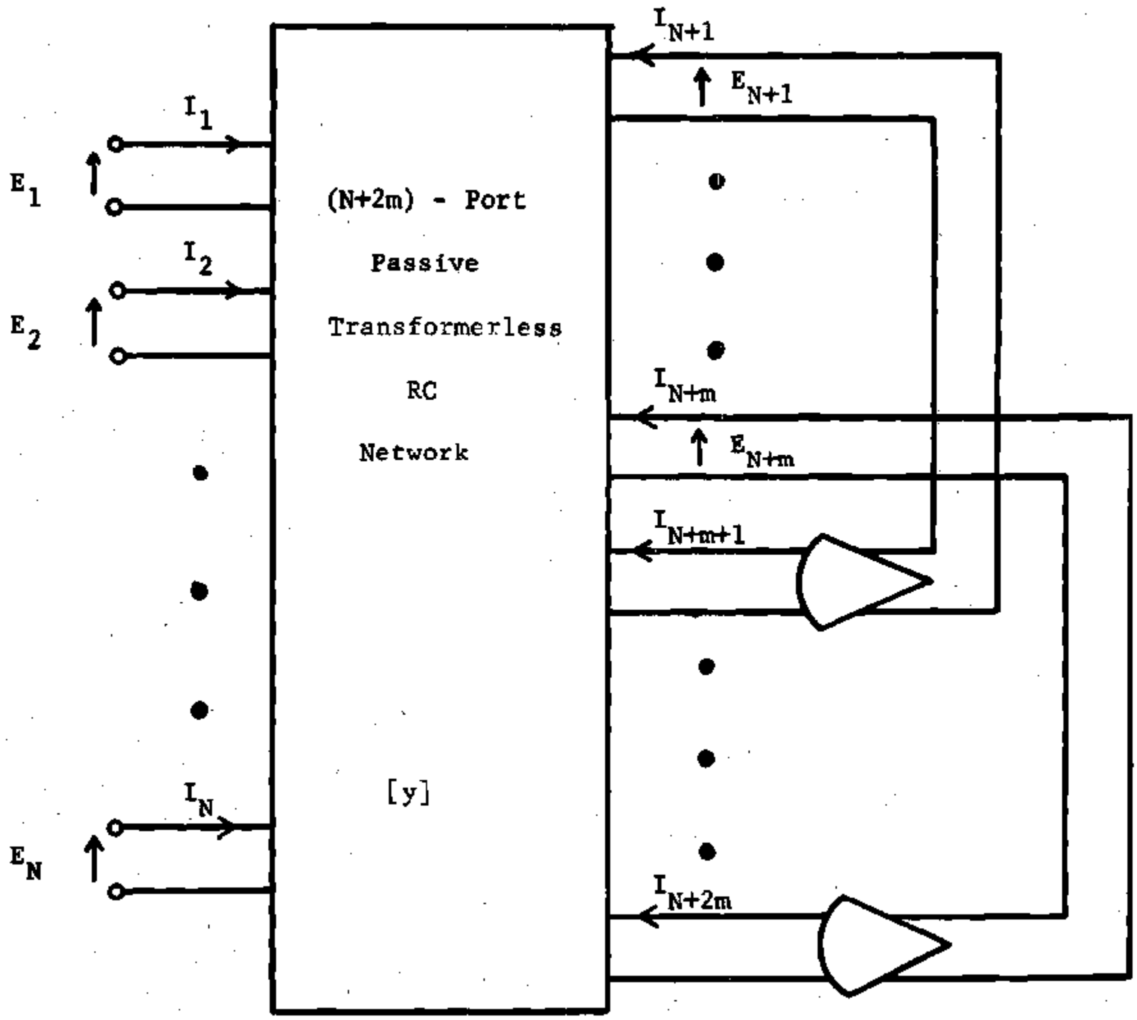

Figure 1. N-Port Active RC Network Containing m Operational Amplifiers.

$$\bar{E}_b = \begin{bmatrix} E_{N+1} \\ E_{N+2} \\ \cdot \\ \cdot \\ \cdot \\ E_{N+m} \end{bmatrix} \qquad \bar{I}_b = \begin{bmatrix} I_{N+1} \\ I_{N+2} \\ \cdot \\ \cdot \\ \cdot \\ I_{N+m} \end{bmatrix}$$

$$\bar{E}_c = \begin{bmatrix} E_{N+m+1} \\ E_{N+m+2} \\ \cdot \\ \cdot \\ \cdot \\ E_{N+2m} \end{bmatrix} \qquad \bar{I}_c = \begin{bmatrix} I_{N+m+1} \\ I_{N+m+2} \\ \cdot \\ \cdot \\ \cdot \\ I_{N+2m} \end{bmatrix}$$

The m operational amplifiers embedded in the N-port network are assumed to be ideal so that the constraints they impose on the system variables are

$$\bar{E}_b = \bar{C}\,\bar{E}_c \tag{2}$$

and

$$\bar{I}_c = 0] \tag{3}$$

The matrix $\bar{C}$ is an m x m diagonal matrix with the ith diagonal element equal to the gain, k_i, of the amplifier connected to port N + i. Partitioning the short-circuit admittance matrix [y] of the (N+2m)-port passive RC network after the Nth and the $(N+m)^{th}$ columns and rows gives

$$\begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{Y}_{13} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{23} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{bmatrix} \begin{bmatrix} \bar{E}_a \\ \bar{E}_b \\ \bar{E}_c \end{bmatrix} \tag{4}$$

where

$$\bar{Y}_{ij} = \bar{Y}^t_{ji}$$

$$i \;\; = 1, 2, 3$$

$$j \;\; = 1, 2, 3 \; .$$

If the constraints imposed by the operational amplifiers are introduced into the above matrix equation, the result is

$$\begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_{11} & (\bar{Y}_{12}\bar{C} + \bar{Y}_{13}) \\ \bar{Y}_{21} & (\bar{Y}_{22}\bar{C} + \bar{Y}_{23}) \\ \bar{Y}_{31} & (\bar{Y}_{32}\bar{C} + \bar{Y}_{33}) \end{bmatrix} \begin{bmatrix} \bar{E}_a \\ \bar{E}_c \end{bmatrix} \tag{5}$$

This equation then yields the two equations

$$\bar{I}_a = \bar{Y}_{11}\bar{E}_a + (\bar{Y}_{12}\bar{C} + \bar{Y}_{13})\bar{E}_c \tag{6}$$

and

$$0] = \bar{Y}_{31}\,\bar{E}_a + (\bar{Y}_{32}\,\bar{C} + \bar{Y}_{33})\,\bar{E}_c \tag{7}$$

Solving Equation (7) for $\bar{E}_c$ as a function of $\bar{E}_a$ gives*

$$\bar{E}_c = -(\bar{Y}_{32}\,\bar{C} + \bar{Y}_{33})^{-1}\,\bar{Y}_{31}\,\bar{E}_a \tag{8}$$

Equation (8) may now be substituted into Equation (6) to yield

$$\bar{I}_a = [\bar{Y}_{11} - (\bar{Y}_{12} + \bar{Y}_{13}\,\bar{C}^{-1})(\bar{Y}_{32} + \bar{Y}_{33}\,\bar{C}^{-1})^{-1}\,\bar{Y}_{31}]\,\bar{E}_a \tag{9}$$

Since the operational amplifiers are ideal, their gains may be assumed to approach infinity. This implies that the diagonal elements of $\bar{C}$ approach infinity or the diagonal elements of $\bar{C}^{-1}$ approach zero. Using this result in Equation (9) gives

$$\bar{I}_a = [\bar{Y}_{11} - \bar{Y}_{12}\,\bar{Y}_{32}^{-1}\,\bar{Y}_{31}]\,\bar{E}_a \tag{10}$$

But if $\bar{I}_a = \bar{Y}\,\bar{E}_a$, then $\bar{Y}$ is the short-circuit admittance matrix for the active N-port network. Thus, from Equation (10)

$$\bar{Y} = \bar{Y}_{11} - \bar{Y}_{12}\,\bar{Y}_{32}^{-1}\,\bar{Y}_{31} \tag{11}$$

*It is assumed that all necessary inverses exist in this chapter and those that follow.

Derivation of a Necessary Condition

A necessary condition of the active network required to realize certain classes of short-circuit admittance matrices will now be derived from Equation (11). Since $\bar{Y}$ may be any matrix of real rational functions in the complex-frequency variable, it may be assumed that $\bar{Y}$ has a pole at $s = s_1$ of order k where s_1 is off the negative-real axis. Multiplying Equation (11) by $(s-s_1)^k$ and evaluating it at $s = s_1$ gives

$$[(s-s_1)^k \bar{Y}]_{s=s_1} = [(s-s_1)^k \bar{Y}_{11}]_{s=s_1} - [(s-s_1)^k \bar{Y}_{12} \bar{Y}_{32}^{-1} \bar{Y}_{31}]_{s=s_1} \quad (12)$$

But $\bar{Y}_{11}$, $\bar{Y}_{12}$, and $\bar{Y}_{31}$ are submatrices of the admittance matrix of a passive RC network and as such may possess only negative-real axis poles. Therefore

$$[(s-s_1)^k \bar{Y}_{11}]_{s=s_1} = [0] \quad (13)$$

and $\bar{Y}_{12}\Big|_{s=s_1}$ and $\bar{Y}_{31}\Big|_{s=s_1}$ are finite. Equation (12) can now be written as

$$[(s-s_1)^k \bar{Y}]_{s=s_1} = -\bar{Y}_{12}\Big|_{s=s_1} [(s-s_1)^k \bar{Y}_{32}^{-1}]_{s=s_1} \bar{Y}_{31}\Big|_{s=s_1} \quad (14)$$

Since the rank of a matrix product cannot exceed the rank of any of its constituent factors[13]

$$\text{rank}\{[(s-s_1)^k \bar{Y}]_{s=s_1}\} \le \text{rank}\ \{\bar{Y}_{31}\Big|_{s=s_1}\}. \quad (15)$$

The rank of $\bar{Y}_{31}\Big|_{s=s_1}$ is limited by the size of the matrix which in this case is N x m. This implies that

$$\text{rank } \{\bar{Y}_{31}\Big|_{s=s_1}\} \leq \min[N,m] \tag{16}$$

In addition to the previous assumption that $\bar{Y}$ possesses a pole at $s = s_1$ of order k, it may be assumed that the rank of this kth order pole is N.* This implies that

$$\text{rank } \{[(s-s_1)^k \bar{Y}]_{s=s_1}\} = N \tag{17}$$

so that Equation (15) becomes, with the use of Equations (16) and (17),

$$N \leq \min [N,m] \tag{18}$$

Obviously, from the above equation

$$m \geq N \tag{19}$$

Note from Figure 1 that m is the number of operational amplifiers embedded in the network. Thus, at least N ideal operational amplifiers are necessary for the realization of any N x N short-circuit admittance matrix provided

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*By "a kth order pole of rank N" it is meant that the matrix of the kth coefficients of the principal parts of the Laurent expansion of $\bar{Y}$ about this kth order pole has rank N.

(a) each element of the matrix is a real rational function in the complex-frequency variable;

(b) the matrix contains a kth order pole of rank N off the negative-real axis; and

(c) the network elements are limited to resistors, capacitors, and ideal operational amplifiers.

Realization Procedure 1

In order to prove the sufficiency of a specific number of ideal operational amplifiers and a transformerless passive RC network for the realization of a prescribed N x N short-circuit admittance matrix, it is required to illustrate a realization procedure for any prescribed matrix. From the previous section, the necessity of N ideal operational amplifiers is known for certain classes of prescribed matrices. Consequently, a synthesis procedure will be developed utilizing an active RC network containing N ideal operational amplifiers. The synthesis procedure will be derived by choosing the short-circuit admittance matrix [y] of the passive RC network such that the admittance matrix given by Equation (11) with m = N is the same as the prescribed short-circuit admittance matrix. Also, [y] must be chosen so as to be realizable as the short-circuit admittance matrix of a transformerless 3N-port passive RC network.

Let the prescribed N x N short-circuit admittance matrix be denoted by

$$\bar{Y} = \frac{[P]}{Q} \tag{20}$$

where [P] is an N x N matrix of polynomials in the complex-frequency variable s, and the polynomial Q, which is a function of s, represents either the common denominator of all elements of $\bar{Y}$ if they are identical, or the least common multiple of all denominators if augmentation is necessary. Choose as appropriate N x N matrix $\bar{Y}_{11}$ which fulfills conditions to be presented below. Employ the notation

$$\bar{Y}_{11} = \frac{[p]}{q} = \frac{[p_{ij}]}{q} \tag{21}$$

where [p] is a matrix of polynomials. Subtracting $\bar{Y}_{11}$ from $\bar{Y}$ gives

$$\bar{Y} - \bar{Y}_{11} = \frac{q\,[P] - Q\,[p]}{qQ} = -\frac{[B]}{q\,Q} \tag{22}$$

where the numerator has been written as the negative of some polynomial matrix [B]. Now assume that $\bar{Y}_{11}$ has been chosen so as to satisfy the following conditions:

(A) $\deg p_{ii} = \deg q = NL_o = M$ where $i = 1,2,\cdots,N$ and $L_o = \max(\deg Q, \deg [P])$;

(B) $\dfrac{p_{ii}}{q}$; $i = 1,2,\cdots,N$; are RC driving-point admittance functions;

(C) $p_{ij} = p_{ji}$ for $i = 1,2,\cdots,N$ and $j = 1,2,\cdots, N$;

(D) if $\bar{Y}_{11}$ is expanded in its Foster form,

$$\bar{Y}_{11} = [A_0] + \sum_{j=1}^{M} [A_j] \frac{s}{s + \sigma_j} \tag{23}$$

where σ_j are the zeros of q, then the coefficient matrices $[A_0]$ and $[A_j]$ satisfy the dominance condition* with the inequality sign;

(E) det [B] contains MN distinct negative-real zeros;

(F) the matric polynomial [B], defined in Equation (22), can be written as the product $[D_1][D_2]$ of two matric polynomials $[D_1]$ and $[D_2]$ of degrees, respectively, M and L_0; and

(G) the matric polynomial $[D_2]$ has the property that det $[D_2]$ has only distinct negative-real zeros different from those of q.

In Appendix 1 it is shown that if $\bar{Y}_{11}$ satisfies Conditions (A) and (B), as is always possible, Condition (E) may always be satisfied by proper choice of p_{ii}. It is shown in Appendix 2 that if Condition (E) is satisfied, Conditions (F) and (G) are always true. Conditions (C) and (D) are easily satisfied by proper choice of the off-diagonal terms of [p]. Thus, it is possible, for any prescribed $\bar{Y}$, to choose $\bar{Y}_{11}$ so as to satisfy the above seven conditions.

Using Condition (F), Equation (22) may be written

---

*A symmetric matrix of real constants is said to be a dominant matrix if each of its main-diagonal elements is not less than the sum of the absolute values of all the other elements in the same row.

$$\bar{Y} - \bar{Y}_{11} = \frac{-[D_1]\,[D_2]}{q\,Q} \tag{23}$$

or introducting Equation (11) with m = N

$$\bar{Y}_{12}\,\bar{Y}_{32}^{-1}\,\bar{Y}_{31} = \frac{[D_1]\,[D_2]}{q\,Q} \tag{24}$$

Rearranging the above equation yields

$$\bar{Y}_{32} = q\,Q\,\bar{Y}_{31}\,[D_2]^{-1}\,[D_1]^{-1}\,\bar{Y}_{12} \tag{25}$$

Now choose

$$\bar{Y}_{12} = \frac{K_1\,[D_1]}{q} \tag{26}$$

and denote $\bar{Y}_{31}$ by

$$\bar{Y}_{31} = \frac{K_2\,[G_{31}]}{q} \tag{27}$$

The constants K_1 and K_2 will be specified later. Choose the polynomial matrix $[G_{31}]$ so that

$$\deg\,[G_{31}] \leq M \tag{28}$$

Hence, Equation (25) becomes

$$\bar{Y}_{32} = \frac{K_1\ K_2\ [G_{31}]\ [D_2]^{-1}\ Q}{q} \tag{29}$$

or

$$\bar{Y}_{32} = \frac{K_1\ K_2\ Q\ [G_{31}]\ \mathrm{adj}\ [D_2]}{q\ \det\ [D_2]} \tag{30}$$

The maximum degree of the numerator of Equation (30) is

$$L_0 + N\ L_0 + (N - 1)\ L_0 = 2\ N\ L_0 \tag{31}$$

where Conditions (A) and (F) have been used along with Equation (28). But, using Condition (F), the degree of the denominator is

$$N\ L_0 + N\ L_0 = 2\ N\ L_0 \tag{32}$$

so that $\bar{Y}_{32}$ in Equation (30) is regular at infinity. Also, from Condition (G), the denominator of $\bar{Y}_{32}$ has only distinct negative-real zeros.

At this point $\bar{Y}_{11}$, $\bar{Y}_{12}$, $\bar{Y}_{31}$, and $\bar{Y}_{32}$ have been specified by Equations (21), (26), (27), and (30), respectively, so that Equation (11) is equivalent to the prescribed admittance matrix $\bar{Y}$. The other restriction on these submatrices is that they must constitute a set which results in a matrix [y] realizable as the short-circuit admittance matrix of a network containing only resistors and capacitors. Conditions sufficient for realization of [y] as a passive RC network without transformers are

(1) the diagonal terms are all RC driving-point admittance functions and

(2) if [y] is expanded in its Foster form, the coefficient matrices are all dominant.

Condition (B) verifies that requirement (1) above is satisfied for y_{ii}, $i = 1,2,\cdots,N$. The remainder of the diagonal terms appear in $\bar{Y}_{22}$ and $\bar{Y}_{33}$, and since these two submatrices are arbitrary as far as Equation (11) is concerned, they may be chosen so that their diagonal elements are RC driving-point admittance functions and thus fulfill requirement (1).

The submatrices $\bar{Y}_{12}$, $\bar{Y}_{31}$, and $\bar{Y}_{32}$ are each multiplied by an unspecified constant which can be made arbitrarily small. Hence, by proper choice of the arbitrary matrices $\bar{Y}_{22}$ and $\bar{Y}_{33}$ and of the constants K_1 and K_2, Condition (D) insures that requirement (2) above may be satisfied. Both of the realizability requirements are now met, and [y] may be synthesized as a 3N-port transformerless passive RC network. Weinberg and Slepian[14] have developed a procedure which may be used for realizing a network satisfying the requirements imposed upon [y]. The resulting network is balanced.

Therefore, for the prescribed matrix $\bar{Y}$, a transformerless passive RC network has been realized so that when it is terminated in N ideal operational amplifiers the admittance matrix $\bar{Y}$ is realized at the remaining ports.

The proofs contained in the previous two sections can now be summarized in the following theorem:

Theorem 1

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless active RC network containing ideal operational amplifiers (a) it is sufficient that the network contains N ideal operational amplifiers; and (b) if the matrix possesses a kth order pole of rank N off the negative-real axis, it is necessary that the network contains N ideal operational amplifiers.

Realization Procedure 2

The realization scheme which has been presented is sufficient to complete the proof of Theorem 1, but it is perhaps not the best method for realizing the prescribed matrix. An alternate realization procedure has been developed and will be presented.

Before the realization procedure may be considered, the desired type of network must be analyzed for its short-circuit admittance matrix. From the necessity proof it is evident that the network must contain at least N ideal operational amplifiers. Hence, the straightforward approach would be to utilize a 3N-port RC network, 2N ports of which may be used for operational amplifier connections. If the N-port active network is analyzed in terms of the 3N(3N+1)/2 admittance parameters of the passive network, rather than in terms of submatrices as before, the equation for the admittance matrix becomes far too involved. Hence, the approach which will be utilized is to first analyze an N-port active RC network containing one operational amplifier and then connect N of these networks in parallel. The admittance matrix of an N-port transformerless active RC network containing N ideal operational amplifiers will thus be found in terms of the $N^2(N+1)/2$ admittance parameters of the N passive networks. The prescribed short-circuit admittance matrix will then be compared with this equation so that the admittance parameters of the passive network can be identified.

Let the short-circuit admittance matrix of the (N+2)-port transformerless passive RC network in Figure 2 be given by

$$[y^{(i)}] = \begin{bmatrix} y_{11}^{(1)} & y_{12}^{(i)} & \cdots & y_{1(N+2)}^{(i)} \\ y_{12}^{(i)} & y_{22}^{(i)} & \cdots & y_{2(N+2)}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{1(N+2)}^{(i)} & y_{2(N+2)}^{(i)} & \cdots & y_{(N+2)(N+2)}^{(i)} \end{bmatrix} \qquad (33)$$

where the superscript indicates the ith of the N networks which will be connected in parallel.

The constraints imposed upon the system variables by the ideal operational amplifier are

$$E_{N+1}^{(i)} = k^{(i)} E_{N+2}^{(i)} \qquad (34)$$

and

$$I_{N+2}^{(i)} = 0 \qquad (35)$$

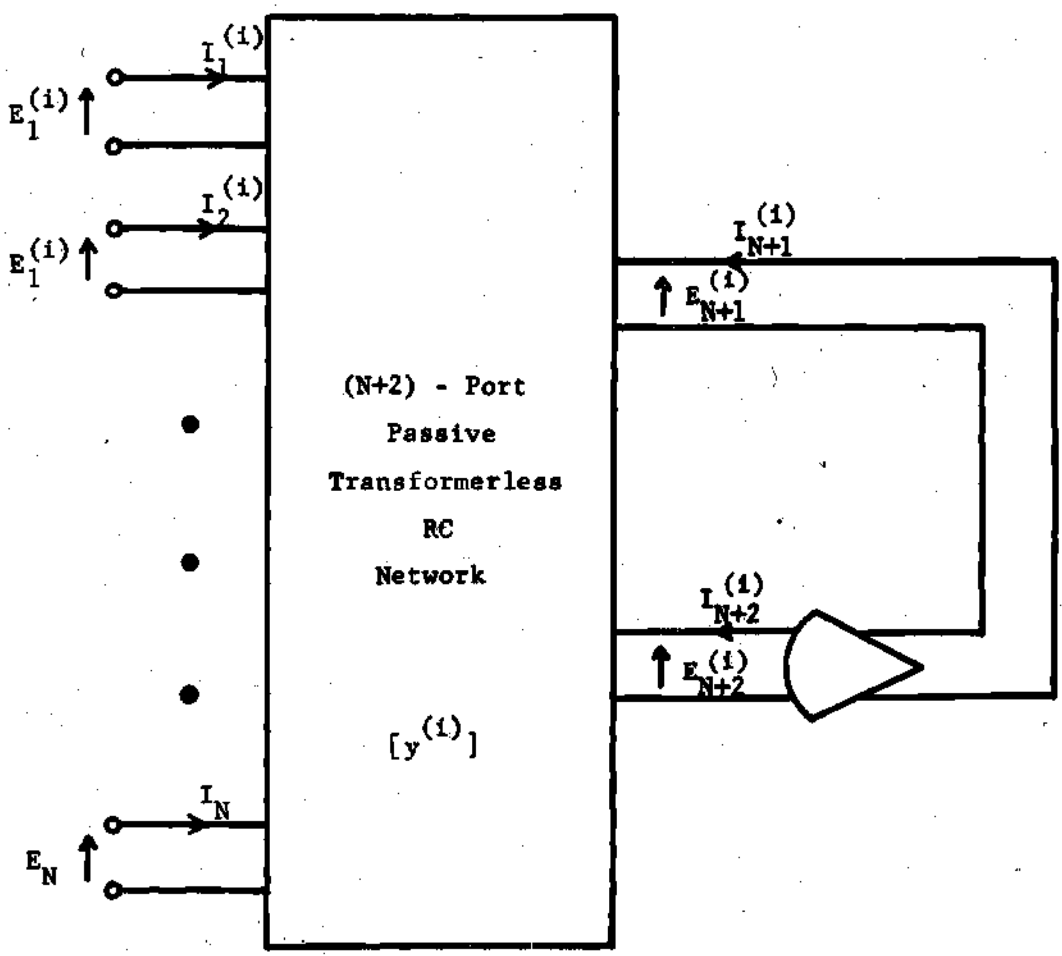

Figure 2. N-Port Active RC Network Containing One Operational Amplifier.

The gain $k^{(i)}$ of the operational amplifier will later be assumed to approach infinity. Using these constraints in conjunction with Equation (33) and the definition of $[y^{(i)}]$ gives

$$\begin{bmatrix} I_1^{(i)} \\ I_2^{(i)} \\ \vdots \\ I_N^{(i)} \end{bmatrix} = \begin{bmatrix} y_{11}^{(i)} & y_{12}^{(i)} & \cdots & y_{1N}^{(i)} \\ y_{12}^{(i)} & y_{22}^{(i)} & \cdots & y_{2N}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{1N}^{(i)} & y_{2N}^{(i)} & \cdots & y_{NN}^{(i)} \end{bmatrix} \begin{bmatrix} E_1^{(i)} \\ E_2^{(i)} \\ \vdots \\ E_N^{(i)} \end{bmatrix} + \begin{bmatrix} k^{(i)} y_{1(N+1)}^{(i)} + y_{1(N+2)}^{(i)} \\ k^{(i)} y_{2(N+1)}^{(i)} + y_{2(N+2)}^{(i)} \\ \vdots \\ k^{(i)} y_{N(N+1)}^{(i)} + y_{N(N+2)}^{(i)} \end{bmatrix} E_{N+2}^{(i)} \tag{36}$$

and

$$y_{1(N+2)}^{(i)} E_1^{(i)} + y_{2(N+2)}^{(i)} E_2^{(i)} + \cdots + y_{N(N+2)}^{(i)} E_N^{(i)} \tag{37}$$

$$+ \left(k^{(i)} y_{(N+1)(N+2)}^{(i)} + y_{(N+2)(N+2)}^{(i)}\right) E_{N+2}^{(i)} = 0$$

Solving Equation (37) for $E_{N+2}^{(i)}$ yields the result

$$E_{N+2}^{(i)} = - \frac{y_{1(N+2)}^{(i)} E_1^{(i)} + y_{2(N+2)}^{(i)} E_2^{(i)} + \cdots + y_{N(N+2)}^{(i)} E_N^{(i)}}{k^{(i)} y_{(N+1)(N+2)}^{(i)} + y_{(N+2)(N+2)}^{(i)}} \tag{38}$$

Equation (38) may now be used to eliminate the variable $E_{N+2}^{(i)}$ from Equation (36) so that

$$\begin{bmatrix} I_1^{(i)} \\ I_2^{(i)} \\ \vdots \\ I_N^{(i)} \end{bmatrix} = \begin{bmatrix} y_{11}^{(i)} & y_{12}^{(i)} & \cdots & y_{1N}^{(i)} \\ y_{12}^{(i)} & y_{22}^{(i)} & \cdots & y_{2N}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{1N}^{(i)} & y_{2N}^{(i)} & \cdots & y_{NN}^{(i)} \end{bmatrix} \begin{bmatrix} E_1^{(i)} \\ E_2^{(i)} \\ \vdots \\ E_N^{(i)} \end{bmatrix} \tag{39}$$

$$- \begin{bmatrix} y_{1(N+1)}^{(i)} + \frac{1}{k^{(i)}} y_{1(N+2)}^{(i)} \\ y_{2(N+1)}^{(i)} + \frac{1}{k^{(i)}} y_{2(N+2)}^{(i)} \\ \vdots \\ y_{N(N+1)}^{(i)} + \frac{1}{k^{(i)}} y_{N(N+2)}^{(i)} \end{bmatrix} \frac{\{y_{1(N+2)}^{(i)} E_1^{(i)} + y_{2(N+2)}^{(i)} E_2^{(i)} + \cdots\ y_{N(N+2)}^{(i)} E_N^{(i)}\}}{y_{(N+1)(N+2)}^{(i)} + \frac{1}{k^{(i)}} y_{(N+2)(N+2)}^{(i)}}$$

Since the operational amplifier is ideal, the gain $k^{(i)}$ may be assumed to approach infinity. If $k^{(i)}$ approaches infinity and the short-circuit admittance matrix of the ith active N-port network is denoted by $[Y^{(i)}]$, Equation (39) yields

$$[y^{(i)}] = \begin{bmatrix} y_{11}^{(i)} & y_{12}^{(i)} & \cdots & y_{1N}^{(i)} \\ y_{12}^{(i)} & y_{22}^{(i)} & \cdots & y_{2N}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{1N}^{(i)} & y_{2N}^{(i)} & \cdots & y_{NN}^{(i)} \end{bmatrix} \tag{40}$$

$$- \frac{1}{y_{(N+1)(N+2)}^{(i)}} = \begin{bmatrix} y_{1(N+1)}^{(i)} y_{1(N+2)}^{(i)} & y_{1(N+1)}^{(i)} y_{2(N+2)}^{(i)} & \cdots & y_{1(N+1)}^{(i)} y_{N(N+2)}^{(i)} \\ y_{2(N+1)}^{(i)} y_{1(N+2)}^{(i)} & y_{2(N+1)}^{(i)} y_{2(N+2)}^{(i)} & \cdots & y_{2(N+1)}^{(i)} y_{N(N+2)}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{N(N+1)}^{(i)} y_{1(N+2)}^{(i)} & y_{N(N+1)}^{(i)} y_{2(N+2)}^{(i)} & \cdots & y_{N(N+1)}^{(i)} y_{N(N+2)}^{(i)} \end{bmatrix}$$

If N such networks are connected in parallel, the new admittance matrix [Y] for the N-port active RC network containing N ideal operational amplifiers is given by*

$$[Y] = \sum_{i=1}^{N} [Y^{(i)}] \tag{41}$$

Using Equation (40) in Equation (41) yields

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*Assume that the network configurations are such that the admittance matrices add without the use of ideal transformers.

$$[Y] = [Y_{11}] - [Y_a][Y_b] \tag{42}$$

where

$$[Y_{11}] = \sum_{i=1}^{N} \begin{bmatrix} y_{11}^{(i)} & y_{12}^{(i)} & \cdots & y_{1N}^{(i)} \\ y_{12}^{(i)} & y_{22}^{(i)} & \cdots & y_{2N}^{(i)} \\ \vdots & \vdots & & \vdots \\ y_{1N}^{(i)} & y_{2N}^{(i)} & \cdots & y_{NN}^{(i)} \end{bmatrix} \tag{43}$$

$$[Y_a] = \begin{bmatrix} y_{1(N+1)}^{(1)} & y_{1(N+1)}^{(2)} & \cdots & y_{1(N+1)}^{(N)} \\ y_{2(N+1)}^{(1)} & y_{2(N+1)}^{(2)} & & y_{2(N+1)}^{(N)} \\ \vdots & \vdots & & \vdots \\ y_{N(N+1)}^{(1)} & y_{N(N+1)}^{(2)} & \cdots & y_{N(N+1)}^{(N)} \end{bmatrix} \tag{44}$$

and

$$[Y_b] = \begin{bmatrix} \frac{y^{(1)}_{1(N+2)}}{y^{(1)}_{(N+1)(N+2)}} & \frac{y^{(1)}_{2(N+2)}}{y^{(1)}_{(N+1)(N+2)}} & \cdots & \frac{y^{(1)}_{N(N+2)}}{y^{(1)}_{(N+1)(N+2)}} \\ \frac{y^{(2)}_{1(N+2)}}{y^{(2)}_{(N+1)(N+2)}} & \frac{y^{(2)}_{2(N+2)}}{y^{(2)}_{(N+1)(N+2)}} & \cdots & \frac{y^{(2)}_{N(N+2)}}{y^{(2)}_{(N+1)(N+2)}} \\ \vdots & \vdots & & \vdots \\ \frac{y^{(N)}_{1(N+2)}}{y^{(N)}_{(N+1)(N+2)}} & \frac{y^{(N)}_{2(N+2)}}{y^{(N)}_{(N+1)(N+2)}} & \cdots & \frac{y^{(N)}_{N(N+2)}}{y^{(N)}_{(N+1)(N+2)}} \end{bmatrix} \quad (45)$$

Now that the desired type of network has been analyzed, the actual realization procedure may be considered. The problem is, given a prescribed short-circuit admittance matrix [Y], how to determine the short-circuit admittance parameters for the N passive networks so as to satisfy Equation (42) and so that the matrices $[y^{(i)}]$ can be realized as transformerless passive RC (N+2)-port networks.

Denote the prescribed N x N short-circuit admittance matrix [Y] by

$$[Y] = \frac{[P]}{Q} \quad (46)$$

as in Equation (20). Let the N x N matrix $[Y_{11}]$ be denoted by

$$[Y_{11}] = \frac{[p]}{q} \tag{47}$$

Subtracting $[Y_{11}]$ from $[Y]$ gives

$$[Y] - [Y_{11}] = \frac{q\,[P] - Q\,[p]}{q\,Q} = -\frac{[B]}{q\,Q} \tag{48}$$

Assume that $[Y_{11}]$ has been chosen so as to satisfy Conditions (A) through (F) where $\bar{Y}_{11}$ is replaced by $[Y_{11}]$. These conditions may always be satisfied by a proper choice of $[Y_{11}]$ as is explained in realization procedure 1.

Condition (F) may be used to rewrite Equation (48) so that

$$[Y] - [Y_{11}] = -\frac{[D_1]\,[D_2]}{q\,Q} \tag{49}$$

This result may be substituted into Equation (42) to yield

$$[Y_a]\,[Y_b] = \frac{[D_1]\,[D_2]}{q\,Q} \tag{50}$$

Constants K_1 and K_2, to be specified later, may be introduced into Equation (50) and the right-hand side rearranged so that the following identification can be accomplished:

$$[Y_a] = \frac{K_1\,[D_1]}{q} \tag{51}$$

and

$$[Y_b] = \frac{\dfrac{K_2\ [D_2]}{q}}{\dfrac{K_1\ K_2\ Q}{q}} \tag{52}$$

Let the following notation be employed:

$$[D_1] = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N} \\ e_{21} & e_{22} & \cdots & e_{2N} \\ \vdots & \vdots & & \vdots \\ e_{N1} & e_{N2} & \cdots & e_{NN} \end{bmatrix} \tag{53}$$

and

$$[D_2] = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{bmatrix} \tag{54}$$

where the e's and h's are polynomials of maximum degrees M and L_0, respectively. Substitution of Equations (53) and (44) into Equation (51) gives

$$\begin{bmatrix} y^{(1)}_{1(N+1)} & y^{(2)}_{1(N+1)} & \cdots & y^{(N)}_{1(N+1)} \\ y^{(1)}_{2(N+1)} & y^{(2)}_{2(N+1)} & \cdots & y^{(N)}_{2(N+1)} \\ \vdots & \vdots & & \vdots \\ y^{(1)}_{N(N+1)} & y^{(2)}_{N(N+1)} & \cdots & y^{(N)}_{N(N+1)} \end{bmatrix} = \begin{bmatrix} \frac{K_1\, e_{11}}{q} & \frac{K_1\, e_{12}}{q} & \cdots & \frac{K_1\, e_{1N}}{q} \\ \frac{K_1\, e_{21}}{q} & \frac{K_1\, e_{22}}{q} & \cdots & \frac{K_1\, e_{2N}}{q} \\ \vdots & \vdots & & \vdots \\ \frac{K_1\, e_{N1}}{q} & \frac{K_1\, e_{N2}}{q} & \cdots & \frac{K_1\, e_{NN}}{q} \end{bmatrix} \tag{55}$$

From this equation N^2 of the y-parameters may easily be identified. Setting corresponding terms of the two matrices equal,

$$y^{(i)}_{j(N+1)} = \frac{K_1\, e_{ji}}{q} \tag{56}$$

where $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,N$.

To obtain a second equation similar to Equation (55), Equations (45) and (54) may be substituted into Equation (52). In order to make the necessary identification, the (N+1), (N+2) y-parameters of all N networks are assumed to be equal. That is

$$y^{(i)}_{(N+1)(N+2)} = y^{(j)}_{(N+1)(N+2)} \tag{57}$$

where $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,N$. Therefore,

$$\frac{1}{y^{(i)}_{(N+1)(N+2)}} \begin{bmatrix} y^{(1)}_{1(N+2)} & y^{(1)}_{2(N+2)} & \cdots & y^{(1)}_{N(N+2)} \\ y^{(2)}_{1(N+2)} & y^{(2)}_{2(N+2)} & \cdots & y^{(2)}_{N(N+2)} \\ \vdots & \vdots & & \vdots \\ y^{(N)}_{1(N+2)} & y^{(N)}_{2(N+2)} & \cdots & y^{(N)}_{N(N+2)} \end{bmatrix} = \tag{58}$$

$$\frac{1}{\frac{K_1 K_2 Q}{q}} \begin{bmatrix} \frac{K_2 h_{11}}{q} & \frac{K_2 h_{12}}{q} & \cdots & \frac{K_2 h_{1N}}{q} \\ \frac{K_2 h_{21}}{q} & \frac{K_2 h_{22}}{q} & \cdots & \frac{K_2 h_{2N}}{q} \\ \vdots & \vdots & & \vdots \\ \frac{K_2 h_{N1}}{q} & \frac{K_2 h_{N2}}{q} & \cdots & \frac{K_2 h_{NN}}{q} \end{bmatrix}$$

and (N^2+1) more of the parameters are specified. Again, corresponding terms of the matrix equation are matched so that the identification below may be made.

$$y^{(i)}_{j(N+2)} = \frac{K_2 h_{ij}}{q} \tag{59}$$

and

$$y^{(i)}_{(N+1)(N+2)} = \frac{K_1 \; K_2 \; Q}{q} \tag{60}$$

where $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,N$.

If it is noted that [p] is the same as $[p_{ij}]$, Equation (47) may be written, with the aid of Equation (43), as

$$\sum_{i=1}^{N} \begin{bmatrix} y^{(i)}_{11} & y^{(i)}_{12} & \cdots & y^{(i)}_{1N} \\ y^{(i)}_{12} & y^{(i)}_{22} & \cdots & y^{(i)}_{2N} \\ \vdots & \vdots & & \vdots \\ y^{(i)}_{1N} & y^{(i)}_{2N} & \cdots & y^{(i)}_{NN} \end{bmatrix} = \begin{bmatrix} \frac{p_{11}}{q} & \frac{p_{12}}{q} & \cdots & \frac{p_{1N}}{q} \\ \frac{p_{12}}{q} & \frac{p_{22}}{q} & \cdots & \frac{p_{2N}}{q} \\ \vdots & \vdots & & \vdots \\ \frac{p_{1N}}{q} & \frac{p_{2N}}{q} & \cdots & \frac{p_{NN}}{q} \end{bmatrix} \tag{61}$$

There are numerous ways in which the y-parameters of Equation (61) may be identified, but one simple identification is

$$y^{(r)}_{ij} = \frac{1}{N} \; \frac{p_{ij}}{q} \tag{62}$$

for $i = 1,2,\ldots,N$; $j = 1,2,\ldots,N$; and $r = 1,2,\ldots,N$.

Disregarding the constant multipliers K_1 and K_2, at this point all of the admittance parameters of the N networks have been determined except $y^{(i)}_{(N+1)(N+1)}$ and $y^{(i)}_{(N+2)(N+2)}$ for $i = 1,2,\ldots,N$. These parameters are not involved in Equation (42), and consequently they may be chosen freely to facilitate realization of the admittance matrices. With the use of Equations (56), (59), (60), and (62), Equation (33) becomes

$$[y^{(i)}] = \frac{1}{q}\begin{bmatrix} \frac{1}{N}p_{11} & \frac{1}{N}p_{12} & \cdots & \frac{1}{N}p_{1N} & K_1 e_{1i} & K_2 h_{i1} \\ \frac{1}{N}p_{12} & \frac{1}{N}p_{22} & \cdots & \frac{1}{N}p_{2N} & K_1 e_{2i} & K_2 h_{i2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{1}{N}p_{1N} & \frac{1}{N}p_{2N} & \cdots & \frac{1}{N}p_{NN} & K_1 e_{Ni} & K_2 h_{iN} \\ K_1 e_{1i} & K_1 e_{2i} & \cdots & K_1 e_{Ni} & C^{(i)}_{(N+1)(N+1)} & K_1 K_2 Q \\ K_2 h_{i1} & K_2 h_{i2} & \cdots & K_2 h_{iN} & K_1 K_2 Q & C^{(i)}_{(N+2)(N+2)} \end{bmatrix} \quad (63)$$

where

$$y^{(i)}_{(N+1)(N+1)} = \frac{C^{(i)}_{(N+1)(N+1)}}{q} \quad (64)$$

and

$$y^{(i)}_{(N+2)(N+2)} = \frac{C^{(i)}_{(N+2)(N+2)}}{q} \quad (65)$$

for $i = 1,2,\ldots,N$.

Even though the y-parameters satisfy Equation (42), $[y^{(i)}]$ for $i = 1,2,\ldots,N$ must still be shown to be realizable as transformerless passive RC networks. The two conditions sufficient for realization of

$[y^{(i)}]$ with a transformerless passive RC network have been given earlier (page 17). Requirement (1) is satisfied since as is stated in Condition (B), p_{ii}/q; where $i = 1,2,\ldots,N$; are RC driving-point admittance functions and $y^{(i)}_{(N+1)(N+1)}$ and $y^{(i)}_{(N+2)(N+2)}$ may be chosen as such.

All elements of $[y^{(i)}]$ possess the same poles, and each element is regular at infinity as can be seen from Conditions (A) and (F). Each off-diagonal term of $[y^{(i)}]$, except those contained in $[Y_{11}]$, is multiplied by a constant multiplier K_1, K_2, or $K_1\ K_2$ yet to be specified. By choosing K_1 and K_2 small, these off-diagonal terms may be made arbitrarily small. But, the coefficient matrices of the Foster expansion of $[Y_{11}]$ satisfy the dominance condition with the inequality sign. Hence, if $[y^{(i)}]$ is expanded in its Foster form, the coefficient matrices can be made dominant merely by choosing K_1 and K_2 sufficiently small; and Condition (2) is satisfied. Realizability of $[y^{(i)}]$ is thus proved, and the technique developed by Weinberg and Slepian[14] may be used for realization. The resulting network is a balanced (N+2)-port transformerless passive RC network.

Each of the matrices $[y^{(i)}]$; $i = 1,2,\ldots,N$; may be realized in the above manner. Since the same constants K_1 and K_2 are involved in each of the matrices $[y^{(i)}]$ where $i = 1,2,\ldots,N$, these constants must be chosen sufficiently small so that each of the N short-circuit admittance matrices has dominant coefficient matrices. Operational amplifiers may now be connected from port N+2 to port N+1 of each of the N networks, and the networks may be connected in parallel at ports 1 through N.* The resulting network is a balanced N-port transformerless active RC network containing N ideal operational amplifiers, and the prescribed short-circuit admittance matrix [Y] is realized at ports 1 through N.

An Example

As an example of realization procedure 1, a 2-port transformerless active RC network containing two ideal operational amplifiers will be found having the prescribed driving-point admittance matrix

$$\bar{Y} = \frac{\begin{bmatrix} s & s - 5 \\ 2 & s + 5 \end{bmatrix}}{s + 5} = \frac{[P]}{Q} \tag{66}$$

Note that this matrix can not be realized using a passive network as is readily seen by its nonreciprocal character. As the first step in the realization, choose

$$\bar{Y}_{11} = \frac{\begin{bmatrix} 50(s + 1) & 1 \\ 1 & 50(s + 1) \end{bmatrix}}{s + 2} = \frac{[p]}{q} \tag{67}$$

The degree M of the polynomial q has not been chosen as large as speci-

---

*Note that ideal transformers are not needed since balance networks are employed.

fied by Condition (A), but the condition is merely sufficient and not necessary. An attempt will be made to use a first-order polynomial for q, but if the procedure fails, the degree will have to be increased. Equation (22) gives

$$[B] = \begin{bmatrix} 49s^2 + 298s + 250 & -s^2 + 4s + 15 \\ -s + 1 & 49s^2 + 293s + 240 \end{bmatrix}$$

Using the procedure in Appendix 2, [B] can be factored so that

$$[B] = [D_1]\,[D_2]$$

where

$$[D_1] = \begin{bmatrix} 49.7602s + 246.866 & 47.4289s + 242.771 \\ -37.2512s - 187.743 & 12.7455s + 62.2611 \end{bmatrix} \tag{68}$$

$$[D_2] = \begin{bmatrix} 0.260113s + 0.251400 & -0.973244s - 0.940644 \\ 0.760228s + 0.77414 & s + 1.01830 \end{bmatrix} \tag{69}$$

Equation (26) yields

$$\bar{Y}_{12} = K_1 \frac{\begin{bmatrix} 49.7602s + 246.866 & 47.4289s + 242.771 \\ -37.2512s - 187.743 & 12.7455s + 62.2611 \end{bmatrix}}{s + 2} \tag{70}$$

Choose $\bar{Y}_{31}$ in Equation (27) so that

$$\bar{Y}_{31} = K_2 \frac{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{s + 2} = K_2 \frac{[G_{13}]}{q} \tag{71}$$

Using Equation (30) and expanding the result in the Foster form yields,

$$\bar{Y}_{32} = K_1K_2 \begin{bmatrix} 2.58664 & 2.38939 \\ -1.96644 & 0.638596 \end{bmatrix} + K_1K_2 \frac{s}{s+2} \begin{bmatrix} 1.45138 & 2.38939 \\ -1.10338 & 0.397443 \end{bmatrix} \tag{72}$$

$$+ K_1K_2 \frac{s}{s + 0.966504} \begin{bmatrix} -4.03842 & 0 \\ 3.07014 & 0 \end{bmatrix} + K_1K_2 \frac{s}{s + 1.01830} \begin{bmatrix} 0 & -3.87645 \\ 0 & -1.036052 \end{bmatrix}$$

The four submatrices involved in Equation (11) have now been determined so that the equation is satisfied for the prescribed matrix given in Equation (66). The submatrices $\bar{Y}_{22}$ and $\bar{Y}_{33}$ can be chosen freely to facilitate realization of [y]. In order to simplify the network, let the off-diagonal terms of $\bar{Y}_{22}$ and $\bar{Y}_{33}$ be zero and use the notation

$$\bar{Y}_{22} = \begin{bmatrix} k_{33}^{(0)} & 0 \\ 0 & k_{44}^{(0)} \end{bmatrix} + \frac{s}{s + 2} \begin{bmatrix} k_{33}^{(1)} & 0 \\ 0 & k_{44}^{(1)} \end{bmatrix} \tag{73}$$

$$+ \frac{s}{s + 0.966504} \begin{bmatrix} k_{33}^{(2)} & 0 \\ 0 & k_{44}^{(2)} \end{bmatrix} + \frac{s}{s + 1.01830} \begin{bmatrix} k_{33}^{(3)} & 0 \\ 0 & k_{44}^{(3)} \end{bmatrix}$$

and

$$\bar{Y}_{33} = \begin{bmatrix} k_{55}^{(0)} & 0 \\ 0 & k_{66}^{(0)} \end{bmatrix} + \frac{s}{s+2} \begin{bmatrix} k_{55}^{(1)} & 0 \\ 0 & k_{66}^{(1)} \end{bmatrix} \tag{74}$$

$$+ \frac{s}{s+0.966504} \begin{bmatrix} k_{55}^{(2)} & 0 \\ 0 & k_{66}^{(2)} \end{bmatrix} + \frac{s}{s+1.01830} \begin{bmatrix} k_{55}^{(3)} & 0 \\ 0 & k_{66}^{(3)} \end{bmatrix}$$

The submatrices may now be used to obtain the short-circuit admittance matrix, [y], of the 6-port transformerless passive RC network. The result is as follows:

$$
[y] = \begin{bmatrix}
25 & 0.5 & 123.4K_1 & 121.4K_1 & 0.5K_2 & 0.5K_2 \\
0.5 & 25 & 93.87K_1 & 31.13K_1 & 0.5K_2 & 0.5K_2 \\
123.4K_1 & 93.87K_1 & k_{33}^{(0)} & 0 & 2.587K_1K_2 & -1.967K_1K_2 \\
121.4K_1 & 31.13K_1 & 0 & k_{44}^{(0)} & 2.389K_1K_2 & -0.6386K_1K_2 \\
0.5K_2 & 0.5K_2 & 2.587K_1K_2 & 2.389K_1K_2 & k_{55}^{(0)} & 0 \\
0.5K_2 & 0.5K_2 & -1.966K_1K_2 & 0.6386K_1K_2 & 0 & k_{66}^{(0)}
\end{bmatrix}
+ \frac{s}{s+2} \begin{bmatrix}
25 & -0.5 & -73.67K_1 & 73.96K_1 & -0.5K_2 & -0.5K_2 \\
-0.5 & 25 & 56.62K_1 & 18.39K_1 & -0.5K_2 & -0.5K_2 \\
-73.67K_1 & 56.62K_1 & k_{33}^{(1)} & 0 & 1.451K_1K_2 & -1.103K_1K_2 \\
73.96K_1 & 18.39K_1 & 0 & k_{44}^{(1)} & 2.389K_1K_2 & 0.3974K_1K_2 \\
-0.5K_2 & -0.5K_2 & 1.451K_1K_2 & 1.487K_1K_2 & k_{55}^{(1)} & 0 \\
-0.5K_2 & -0.5K_2 & -1.103K_1K_2 & 0.3974K_1K_2 & 0 & k_{66}^{(1)}
\end{bmatrix}
+ \frac{s}{s+0.9665} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{33}^{(2)} & 0 & -4.038K_1K_2 & 3.070K_1K_2 \\
0 & 0 & 0 & k_{44}^{(2)} & 0 & 0 \\
0 & 0 & -4.038K_1K_2 & 0 & k_{55}^{(2)} & 0 \\
0 & 0 & 3.070K_1K_2 & 0 & 0 & k_{66}^{(2)}
\end{bmatrix}
+ \frac{s}{s+1.018} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{33}^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & k_{44}^{(3)} & -3.876K_1K_2 & -1.036K_1K_2 \\
0 & 0 & 0 & -3.876K_1K_2 & k_{55}^{(3)} & 0 \\
0 & 0 & 0 & -1.036K_1K_2 & 0 & k_{66}^{(3)}
\end{bmatrix}
\quad (75)
$$

Values will not be assigned to the k's; they will be allowed to have values such that rows 3, 4, 5, and 6 of the coefficient matrices satisfy the dominance condition with the equal sign. The constants K_1 and K_2 must be chosen such that rows 1 and 2 of the coefficient matrices satisfy the dominance condition. One such choice is $K_1 = 0.05$ and $K_2 = 1$, but it is advantageous to attempt to force either rows 1 or 2 or both to satisfy the dominance condition with equality.

The procedure developed by Weinberg and Slepian[14] can now be used to realize each matrix of Equation (75) independently with a balanced transformerless passive RC network. The four networks may then be connected in parallel to yield the complete passive network. Connection of ideal operational amplifiers from port 6 to port 4 and from port 5 to port 3, completes the realization.

CHAPTER III

N x N SHORT-CIRCUIT ADMITTANCE MATRIX SYNTHESIS USING A GROUNDED NETWORK

The balanced networks resulting from the previously described realization procedures suffer from numerous practical disadvantages. Consequently, a realization scheme which yields a grounded network would be extremely desirable for many applications. In this chapter the realization of an N x N short-circuit admittance matrix with a transformerless grounded active RC network containing ideal operational amplifiers will be considered.

The short-circuit admittance matrix for a grounded network is much more restricted than that of one with a balanced configuration. This implies that the identification of the admittance parameters in the grounded realization procedure will be more difficult because of the additional requirements which the parameters must satisfy. In the previous chapter, N differential input operational amplifiers were employed in the balanced realization, but in order to accomplish the realization with a grounded network, 2N single ended operational amplifiers will be utilized. From a practical standpoint, this increase in the number of operational amplifiers is not as serious as it appears because of the numerous advantages offered by the grounded network.

The method which will be employed in the development of a realization procedure is similar to the one used previously. First, the desired type of network will be analyzed so as to determine its short-circuit admittance matrix in terms of the admittance parameters of the passive portion of the network. The prescribed short-circuit admittance matrix will then be compared with the derived matrix equation. This comparison will be used to identify the admittance parameters of the passive network so as to satisfy the equation and yield a matrix which is realizable as a transformerless grounded passive RC network.

Analysis of the Network

Consider Figure 3 which is a grounded N-port transformerless active RC network containing 2N ideal operational amplifiers. Let the short-circuit admittance matrix [y] of the 5N-port transformerless passive RC network be partitioned into 25 N x N submatrices so that

$$\begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \\ \bar{I}_d \\ \bar{I}_e \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{Y}_{13} & \bar{Y}_{14} & \bar{Y}_{15} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{23} & \bar{Y}_{24} & \bar{Y}_{25} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} & \bar{Y}_{34} & \bar{Y}_{35} \\ \bar{Y}_{41} & \bar{Y}_{42} & \bar{Y}_{43} & \bar{Y}_{44} & \bar{Y}_{45} \\ \bar{Y}_{51} & \bar{Y}_{52} & \bar{Y}_{53} & \bar{Y}_{54} & \bar{Y}_{55} \end{bmatrix} \begin{bmatrix} \bar{E}_a \\ \bar{E}_b \\ \bar{E}_c \\ \bar{E}_d \\ \bar{E}_e \end{bmatrix} \tag{76}$$

where $\bar{Y}_{ij} = \bar{Y}_{ji}^t$ for $i = 1,2,3,4,5$ and $j = 1,2,3,4,5$ and the $\bar{I}$'s and $\bar{E}$'s are column matrices of port currents and voltages, respectively, consisting of N variables each. That is, $\bar{I}_a$ consists of the currents at the first N ports and $\bar{I}_b$ consists of the currents at ports N + 1 through 2N. The constraints imposed on the system variables by the

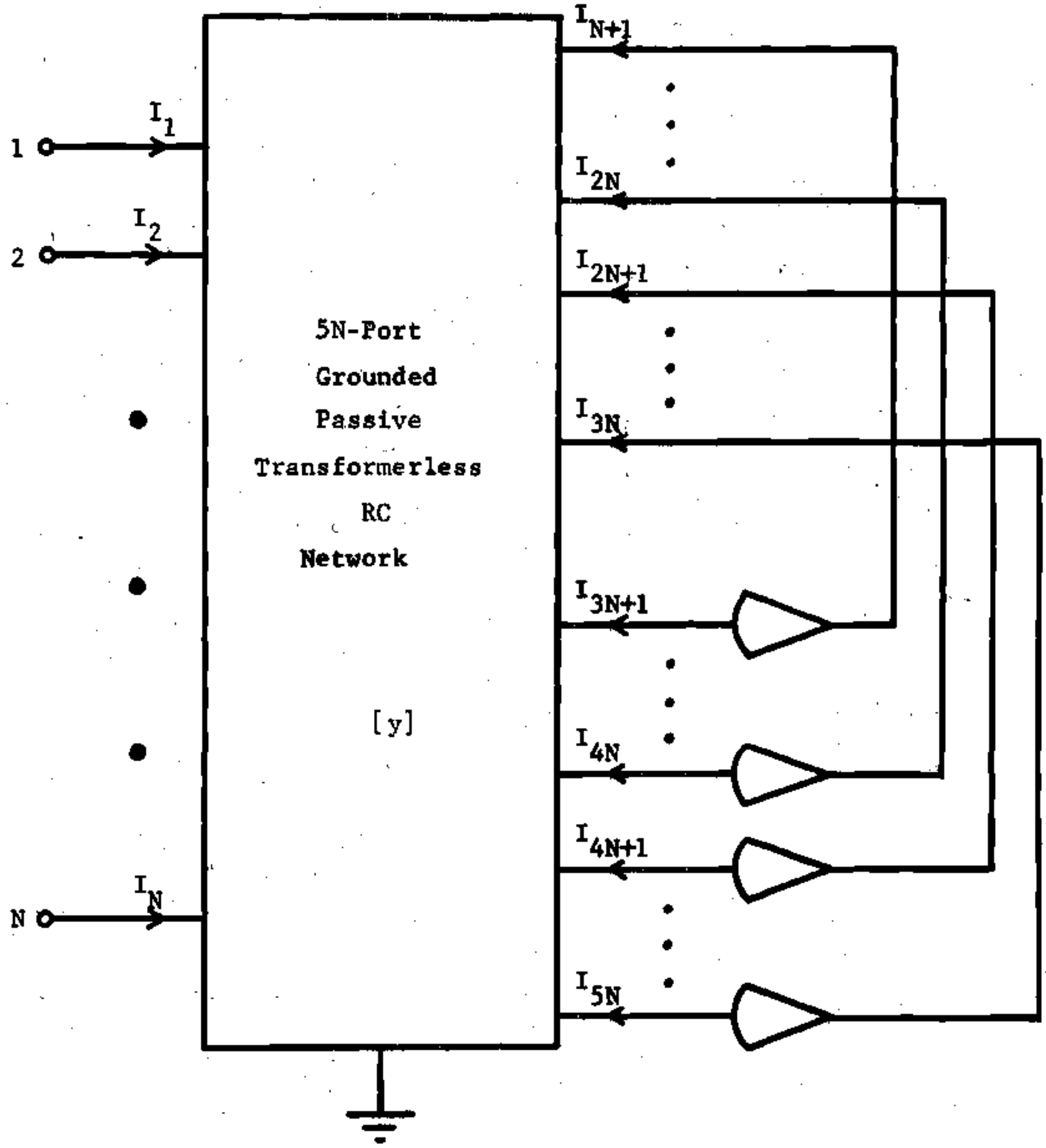


Figure 3 . Grounded N-Port Active RC Network Containing 2N Operational Amplifiers.

ideal operational amplifiers are

$$\bar{I}_d = \bar{I}_e = 0] \tag{77}$$

$$\bar{E}_b = \bar{k}_d \bar{E}_d \tag{78}$$

and

$$\bar{E}_c = \bar{k}_e \bar{E}_e \tag{79}$$

where $\bar{k}_d$ and $\bar{k}_e$ are N x N diagonal matrices with the amplifier gains as diagonal elements. The ith diagonal elements of $\bar{k}_d$ and $\bar{k}_e$ are the gains of the amplifiers connected between ports 3N + i and N + i and ports 4N + i and 2N + i, respectively.

If Equations (77), (78), and (79) are substituted into Equation (76), it becomes

$$\begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \\ 0] \\ 0] \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11} & (\bar{Y}_{12} \bar{k}_d + \bar{Y}_{14}) & (\bar{Y}_{13} \bar{k}_e + \bar{Y}_{15}) \\ \bar{Y}_{21} & (\bar{Y}_{22} \bar{k}_d + \bar{Y}_{24}) & (\bar{Y}_{23} \bar{k}_e + \bar{Y}_{25}) \\ \bar{Y}_{31} & (\bar{Y}_{32} \bar{k}_d + \bar{Y}_{34}) & (\bar{Y}_{33} \bar{k}_e + \bar{Y}_{35}) \\ \bar{Y}_{41} & (\bar{Y}_{42} \bar{k}_d + \bar{Y}_{44}) & (\bar{Y}_{43} \bar{k}_e + \bar{Y}_{45}) \\ \bar{Y}_{51} & (\bar{Y}_{52} \bar{k}_d + \bar{Y}_{54}) & (\bar{Y}_{53} \bar{k}_e + \bar{Y}_{55}) \end{bmatrix} \begin{bmatrix} \bar{E}_a \\ \bar{E}_d \\ \bar{E}_e \end{bmatrix} \tag{80}$$

Rows 4 and 5 of the above matrix equation may be rearranged so that

$$(\bar{Y}_{42}\ \bar{k}_d + \bar{Y}_{44})\ \bar{E}_d + (\bar{Y}_{43}\ \bar{k}_e + \bar{Y}_{45})\ \bar{E}_e = -\ \bar{Y}_{41}\ \bar{E}_a \tag{81}$$

and

$$(\bar{Y}_{52}\ \bar{k}_d + \bar{Y}_{54})\ \bar{E}_d + (\bar{Y}_{53}\ \bar{k}_e + \bar{Y}_{55})\ \bar{E}_e = -\ \bar{Y}_{51}\ \bar{E}_a \tag{82}$$

Solving these two equations for $\bar{E}_d$ and $\bar{E}_e$ yields

$$\bar{E}_d = [(\bar{Y}_{43}\bar{k}_e + \bar{Y}_{45})^{-1}(\bar{Y}_{42}\bar{k}_d + \bar{Y}_{44}) - (\bar{Y}_{53}\bar{k}_e + \bar{Y}_{55})^{-1}(\bar{Y}_{52}\bar{k}_d + \bar{Y}_{54})]^{-1} \cdot [(\bar{Y}_{53}\bar{k}_e + \bar{Y}_{55})^{-1}\ \bar{Y}_{51} - (\bar{Y}_{43}\bar{k}_e + \bar{Y}_{45})^{-1}\ \bar{Y}_{41}]\ \bar{E}_a \tag{83}$$

and

$$\bar{E}_e = [(\bar{Y}_{42}\bar{k}_d + \bar{Y}_{44})^{-1}(\bar{Y}_{43}\bar{k}_e + \bar{Y}_{45}) - (\bar{Y}_{52}\bar{k}_d + \bar{Y}_{54})^{-1}(\bar{Y}_{53}\bar{k}_e + \bar{Y}_{55})]^{-1} \cdot [(\bar{Y}_{52}\bar{k}_d + \bar{Y}_{54})^{-1}\ \bar{Y}_{51} - (\bar{Y}_{42}\bar{k}_d + \bar{Y}_{44})^{-1}\ \bar{Y}_{41}]\ \bar{E}_a \tag{84}$$

Row 1 of Equation (80) can be written as

$$\bar{I}_a = \bar{Y}_{11}\ \bar{E}_a + (\bar{Y}_{12} + \bar{Y}_{14}\ \bar{k}_d^{\ -1})\bar{k}_d\ \bar{E}_d + (\bar{Y}_{13} + \bar{Y}_{15}\ \bar{k}_e^{\ -1})\ \bar{k}_e\ \bar{E}_e \tag{85}$$

Note that the short-circuit admittance matrix $\bar{Y}$ of the active N-port network in Figure 3 is defined by

$$\bar{I}_a = \bar{Y}\,\bar{E}_a \tag{86}$$

so that substitution of Equations (83) and (84) into Equation (85) yields

$$\begin{aligned}\bar{Y} = \bar{Y}_{11} &+ \{[\bar{Y}_{12} + \bar{Y}_{14}\bar{k}_d^{-1}][(\bar{Y}_{43} + \bar{Y}_{45}\bar{k}_e^{-1})^{-1}(\bar{Y}_{42} + \bar{Y}_{44}\bar{k}_d^{-1}) \\ &- (\bar{Y}_{53} + \bar{Y}_{55}\bar{k}_e^{-1})^{-1}(\bar{Y}_{52} + \bar{Y}_{54}\bar{k}_d^{-1})]^{-1}[(\bar{Y}_{53} + \bar{Y}_{55}\bar{k}_e^{-1})^{-1}\bar{Y}_{51} \\ &- (\bar{Y}_{43} + \bar{Y}_{45}\bar{k}_e^{-1})^{-1}\bar{Y}_{41}]\} + \{[\bar{Y}_{13} + \bar{Y}_{15}\bar{k}_e^{-1}][(\bar{Y}_{42} + \bar{Y}_{44}\bar{k}_d^{-1})^{-1} \\ &\cdot (\bar{Y}_{43} + \bar{Y}_{45}\bar{k}_e^{-1}) - (\bar{Y}_{52} + \bar{Y}_{54}\bar{k}_d^{-1})^{-1}(\bar{Y}_{53} + \bar{Y}_{55}\bar{k}_e^{-1})]^{-1} \\ &\cdot [(\bar{Y}_{52} + \bar{Y}_{54}\bar{k}_d^{-1})^{-1}\bar{Y}_{51} - (\bar{Y}_{42} + \bar{Y}_{44}\bar{k}_d^{-1})^{-1}\bar{Y}_{41}]\}\end{aligned} \tag{87}$$

Since the operational amplifiers are ideal, their gains may be assumed to approach infinity. This implies that the diagonal terms of the matrices $\bar{k}_d$ and $\bar{k}_e$ approach infinity or the diagonal terms of $\bar{k}_d^{-1}$ and $\bar{k}_e^{-1}$ approach zero. Using this result in Equation (87) yields

$$\begin{aligned}\bar{Y} = \bar{Y}_{11} &+ \bar{Y}_{12}(\bar{Y}_{43}^{-1}\bar{Y}_{42} - \bar{Y}_{53}^{-1}\bar{Y}_{52})^{-1}(\bar{Y}_{53}^{-1}\bar{Y}_{51} - \bar{Y}_{43}^{-1}\bar{Y}_{41}) \\ &+ \bar{Y}_{13}(\bar{Y}_{42}^{-1}\bar{Y}_{43} + \bar{Y}_{52}^{-1}\bar{Y}_{53})^{-1}(\bar{Y}_{52}^{-1}\bar{Y}_{51} - \bar{Y}_{42}^{-1}\bar{Y}_{41})\end{aligned} \tag{88}$$

If the assumptions

$$\bar{Y}_{13} = [0] \tag{89}$$

and

$$\bar{Y}_{43} = \bar{Y}_{53} \tag{90}$$

are made, Equation (88) reduces to

$$\bar{Y} = \bar{Y}_{11} + \bar{Y}_{12} (\bar{Y}_{42} - \bar{Y}_{52})^{-1} \cdot (\bar{Y}_{51} - \bar{Y}_{41}) \tag{91}$$

which is the short-circuit admittance matrix of the N-port transformerless grounded active RC network containing 2N ideal operational amplifiers. Note that Equation (91) is true only if the submatrices satisfy the conditions of Equations (89) and (90).

Realization Procedure

Now that the analysis of the desired type of network has been performed, the actual realization procedure may be considered. Let the prescribed N x N short-circuit admittance matrix be denoted by

$$\bar{Y} = \frac{[P]}{Q} \tag{92}$$

where [P] is a matrix of polynomials in the complex-frequency variable s, and Q, a polynomial in s, is the common denominator of all elements of $\bar{Y}$ or the least common multiple of all denominators. Choose an appropriate N x N short-circuit admittance matrix $\bar{Y}_{11}$ which fulfills conditions to be presented below. Employ the notation

$$\bar{Y}_{11} = \frac{[p]}{q} \tag{93}$$

where [p] is a matrix of polynomials and q is the common denominator of all elements of $\bar{Y}_{11}$. Equations (92) and (93) may be subtracted to give

$$\bar{Y} - \bar{Y}_{11} = \frac{q\ [P] - Q\ [p]}{q\ Q} = \frac{[B]}{q\ Q} \tag{94}$$

The conditions which $\bar{Y}_{11}$ must fulfill are the following:

(A) $\deg p_{ij} = \deg q = NL_0 = M$ where $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$ and $L_0 = \mathrm{Max}\ (\deg Q, \deg [P])$;

(B) the diagonal and off-diagonal terms are, respectively, positive and negative RC driving-point admittance functions;

(C) the matrix is symmetric;

(D) if $\bar{Y}_{11}$ is expanded in its Foster form as

$$\bar{Y}_{11} = [A_0] + \sum_{j=1}^{M} [A_j] \frac{s}{s + \alpha_j}$$

where α_j are the zeros of q, then the coefficient matrices $[A_0]$ and $[A_j]$ satisfy the dominance condition with the inequality sign;

(E) det [B] contains MN distinct negative-real zeros;

(F) the matric polynomial [B] can be written as the product $[D_b][D_a]$ where $[D_a]$ is of degree M and each term of $[D_b]$ is of degree L_0; and

(G) the matric polynomial $[D_b]$ has the property that det $[D_b]$ has only distinct negative-real zeros different from those of q.

If $\bar{Y}_{11}$ satisfies Conditions (A) and (B), as is always possible, it is shown in Appendix 1 that Condition (E) can always be satisfied by proper choice of p_{ii} for $i = 1,2,\ldots,N$. In Appendix 2 it is shown that if Condition (E) is satisfied, $[B]^t$ can be factored so that

$$[B]^t = [D_1][D_2] \tag{95}$$

where $[D_1]$ possesses the properties required of $[D_a]$ in Conditions (F) and $[D_2]$ possesses those required of $[D_b]$ in Conditions (F) and (G). Taking the transpose of Equation (95) gives

$$[B] = [D_2]^t\,[D_1]^t \tag{96}$$

Now define

$$[D_b] = [D_2]^t \tag{97}$$

and

$$[D_a] = [D_1]^t \tag{98}$$

so that

$$[B] = [D_b][D_a] \tag{99}$$

Note that the requirements imposed upon $[D_a]$ and $[D_b]$ are such that if $[D_1]$ and $[D_2]$ satisfy these requirements, then $[D_1]^t$ and $[D_2]^t$ will also satisfy them. Therefore Equation (99) is the required factorization since $[D_2]^t$ and $[D_1]^t$ possess the properties required of $[D_b]$ and $[D_a]$, respectively. Thus, Conditions (F) and (G) can always be satisfied. Conditions (C) and (D) are easily satisfied by proper choice of the off-diagonal terms of $[p]$. Consequently, it is possible for any prescribed $\bar{Y}$ to choose $\bar{Y}_{11}$ so that all seven conditions are satisfied.

Equations (91), (94), and (99) yield the result

$$\bar{Y}_{12}\ (\bar{Y}_{42} - \bar{Y}_{52})^{-1}\ (\bar{Y}_{51} - \bar{Y}_{41}) = \frac{[D_b][D_a]}{q\ Q} \tag{100}$$

Solving the above equation for $\bar{Y}_{42} - \bar{Y}_{52}$ yields

$$\bar{Y}_{42} - \bar{Y}_{52} = q\ Q\ (\bar{Y}_{51} - \bar{Y}_{41})\ [D_a]^{-1}\ [D_b]^{-1}\ \bar{Y}_{12} \tag{101}$$

Choose

$$\bar{Y}_{51} - \bar{Y}_{41} = K_1\ \frac{[D_a]}{q} \tag{102}$$

and denote $\bar{Y}_{12}$ by

$$\bar{Y}_{12} = \frac{K_2 \, [N_{12}]}{q} \tag{103}$$

where K_1 and K_2 are constants to be specified later. Choose $[N_{12}]$ as a polynomial matrix so that each element of $\bar{Y}_{12}$ is a negative-RC driving-point admittance function and

$$\deg \, [N_{12}] = M \tag{104}$$

Equation (101) then becomes

$$\bar{Y}_{42} - \bar{Y}_{52} = \frac{K_1 \, K_2 \, Q \, (\mathrm{adj}[D_b]) \, [N_{12}]}{q \, \det \, [D_b]} \tag{105}$$

By Condition (G), det $[D_b]$ has only distinct negative-real zeros different from those of q. Therefore

$$q \, \det \, [D_b] = C_1 \prod_{m=1}^{T} (s + \gamma_m) \tag{106}$$

where the γ's are real nonzero positive distinct numbers, C_1 is a constant multiplier, and T is the order of the polynomial q det $[D_b]$.

To show that Equation (105) is regular at infinity, the degree of the numerator must be shown less than or equal to that of the denominator for each element of the matrix. This implies that

$$\deg Q + \deg(\text{adj}\,[D_b]) + \deg\,[N_{12}] \le \deg q + \deg(\det\,[D_b]) \tag{107}$$

But

$$\deg(\text{adj}\,[D_b]) = (N-1)\,L_0 \tag{108}$$

and

$$\deg(\det\,[D_b]) = NL_0 \tag{109}$$

Thus, Equation (107) reduces to

$$\deg Q \le L_0 \tag{110}$$

Using the definition of L_0 given in Condition (A), Equation (110) becomes

$$\deg Q \le \text{Max}\,(\deg\,[P],\, \deg Q) \tag{111}$$

which is obviously true. Thus, Equation (105) is regular at infinity and can be written in its Foster expansion as

$$\bar{Y}_{42} - \bar{Y}_{52} = K_1\, K_2 \sum_{m=0}^{T} \bar{F}_m \frac{s}{s + \gamma_m} \tag{112}$$

where $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_T$ and the $\bar{F}_m$ are real coefficient matrices. Rewriting the above equation in another manner,

$$\bar{Y}_{42} - \bar{Y}_{52} = K_1 K_2 \sum_{m=0}^{T} \bar{G}_m \frac{s}{s + \gamma_m} - K_1 K_2 \sum_{m=0}^{T} \bar{H}_m \frac{s}{s + \gamma_m} \tag{113}$$

where each element in the coefficient matrices $\bar{G}_m$ and $\bar{H}_m$ is real and non-negative. The following identification of parameters can now be made:

$$\bar{Y}_{42} = - K_1 K_2 \sum_{m=0}^{T} \bar{H}_m \frac{s}{s + \gamma_m} \tag{114}$$

and

$$\bar{Y}_{52} = - K_1 K_2 \sum_{m=0}^{T} \bar{G}_m \frac{s}{s + \gamma_m} \tag{115}$$

Note that each element of the submatrices $\bar{Y}_{42}$ and $\bar{Y}_{52}$ is a negative-RC driving-point admittance function.

Equation (102) can also be expanded in a similar manner since from Condition (F), $[D_a]$ is of degree M so that $[D_a]/q$ is regular at infinity. Thus, if

$$q = C_2 \prod_{v=1}^{M} (s + \alpha_v) \tag{116}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_M$ and C_2 is a constant multiplier, then

$$\bar{Y}_{51} - \bar{Y}_{41} = K_1 \sum_{v=0}^{M} \bar{A}_v \frac{s}{s + \alpha_v} - K_1 \sum_{v=0}^{M} \bar{E}_v \frac{s}{s + \alpha_v} \tag{117}$$

where $\alpha_0 = 0$ and $\bar{A}_v$ and $\bar{E}_v$ are coefficient matrices, all of which have real non-negative elements. Make the identification

$$\bar{Y}_{51} = - K_1 \sum_{v=0}^{M} \bar{E}_v \frac{s}{s + \alpha_v} \tag{118}$$

and

$$\bar{Y}_{41} = - K_1 \sum_{v=0}^{M} \bar{A}_v \frac{s}{s + \alpha_v} \tag{119}$$

An examination of the above two equations reveals that the submatrices $\bar{Y}_{51}$ and $\bar{Y}_{41}$ have all elements negative-RC driving-point admittance functions.

All the submatrices involved in Equation (91) have now been specified so as to satisfy this equation, and all that remains in the realization is to show that the set of submatrices forms a short-circuit admittance matrix [y] which is realizable as a transformerless grounded passive RC network. It is well known[15] that the necessary and sufficient conditions for the realization of [y] as a (5N+1)-terminal network of two-terminal impedances with common reference node and no internal nodes are as follows:

1. The diagonal and off-diagonal terms are respectively positive and negative RC driving-point admittance functions; and

2. if [y] is expanded in its Foster expansion, all the coefficient matrices are dominant.

From Condition (B) and Equations (93), (114), (115), (118), and (119) it is seen that Condition (1) above is satisfied for all the parameters which have been specified. Since the undetermined submatrices $\bar{Y}_{22}$, $\bar{Y}_{33}$, $\bar{Y}_{44}$, $\bar{Y}_{55}$, $\bar{Y}_{23}$, $\bar{Y}_{35}$, $\bar{Y}_{34}$, and $\bar{Y}_{45}$ may be chosen freely to facilitate realization of [y], they may be chosen so as to satisfy Condition (1).

An examination of the procedure reveals that all the submatrices which have been specified, except $\bar{Y}_{11}$, are multiplied by an arbitrary constant K_1, K_2, or K_1K_2. The terms of all the specified submatrices except $\bar{Y}_{11}$ can thus be made as small as necessary merely by choosing K_1 and K_2 small. Condition (D) states that $\bar{Y}_{11}$ satisfies the dominance condition with the inequality sign. Therefore, proper choice of the undetermined submatrices along with the choice of K_1 and K_2 can be used to insure that the coefficient matrices are dominant so as to satisfy Condition (2). All the requirements for realizability of [y] with the desired type of network have been met, and the passive network can be realized from the admittance matrix. A procedure which may be used to realize the 5N x 5N short-circuit admittance matrix [y] as a transformerless passive (5N + 1)-terminal network of two-terminal impedances with common reference node and no internal nodes can be found in reference (15). If 2N ideal operational amplifiers are connected to ports N+1 through 5N of the resulting network, the prescribed short-circuit admittance matrix is ralized at the first N ports.

The realization procedure which has been presented in this chapter can be summarized in the following theorem:

<u>Theorem 2</u>

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless grounded active RC network containing ideal operational amplifiers, it is sufficient that the active network contains 2N operational amplifiers.

An Example

To illustrate the grounded realization procedure, a 2-port network will be found having the prescribed short-circuit admittance matrix

$$\bar{Y} = \frac{1}{s+1}\begin{bmatrix} s & s+1 \\ 1 & s+3 \end{bmatrix} \tag{120}$$

First, a two-by-two short-circuit admittance matrix will be chosen for $\bar{Y}_{11}$. Let

$$\bar{Y}_{11} = \frac{1}{s+4}\begin{bmatrix} 50(s+2) & -(s+2) \\ -(s+3) & 50(s+3) \end{bmatrix} \tag{121}$$

Subtracting $\bar{Y}_{11}$ from $\bar{Y}$ gives

$$\bar{Y} - \bar{Y}_{11} = \frac{\begin{bmatrix} -49s^2 - 146s - 100 & 2s^2 + 8s + 6 \\ s^2 + 5s + 7 & -49s^2 - 193s - 138 \end{bmatrix}}{(s+1)(s+4)}$$

$$= \frac{[B]}{q\ Q}$$

Using the matrix factorization procedure in Appendix 2,

$$[B] = \begin{bmatrix} -49.0422s - 52.3262 & 2s + 2 \\ 2.03299s + 5.17858 & -49s - 46 \end{bmatrix} \begin{bmatrix} s + 1.91351 & 0 \\ 0.0210814s + 0.0632442 & s+3 \end{bmatrix}$$

$$= [D_b][D_a] \tag{122}$$

From Equation (102)

$$\bar{Y}_{51} - \bar{Y}_{41} = K_1 \frac{\begin{bmatrix} s + 1.91351 & 0 \\ 0.0210814s + 0.0632442 & s + 3 \end{bmatrix}}{s + 4}$$

After expanding the above equation in the Foster form, the following identification can be made:

$$\bar{Y}_{41} = -K_1 \begin{bmatrix} 0.47838 & 0 \\ 0.015811 & 0.75 \end{bmatrix} - K_1 \frac{s}{s + 4} \begin{bmatrix} 0.52162 & 0 \\ 0.0052704 & 0.25 \end{bmatrix} \tag{123}$$

and

$$\bar{Y}_{51} = [0] \tag{124}$$

Choose

$$\bar{Y}_{12} = K_2 \frac{1}{s + 4} \begin{bmatrix} -s - 1 & -s - 1 \\ -s - 1 & -s - 1 \end{bmatrix} \tag{125}$$

Equations (104), (114), and (115) may be used to obtain

$$\bar{Y}_{42} = -K_1\ K_2 \frac{s}{s + 0.93746} \begin{bmatrix} 8.0516 \times 10^{-7} & 8.0516 \times 10^{-7} \\ 4.2330 \times 10^{-5} & 4.2330 \times 10^{-5} \end{bmatrix} \tag{126}$$

$$-K_1\ K_2 \frac{s}{s + 1.06567} \begin{bmatrix} 2.8821 \times 10^{-5} & 2.8821 \times 10^{-5} \\ 0 & 0 \end{bmatrix}$$

and

$$\bar{Y}_{52} = -K_1\ K_2 \begin{bmatrix} 5.0070 \times 10^{-3} & 5.0070 \times 10^{-3} \\ 5.9984 \times 10^{-3} & 5.9984 \times 10^{-3} \end{bmatrix} \tag{127}$$

$$-\ K_1\ K_2 \frac{s}{s + 4} \begin{bmatrix} 1.6281 \times 10^{-2} & 1.6281 \times 10^{-2} \\ 1.5321 \times 10^{-2} & 1.5321 \times 10^{-2} \end{bmatrix}$$

$$-K_1\ K_2 \frac{s}{s + 1.06567} \begin{bmatrix} 0 & 0 \\ 1.34123 \times 10^{-5} & 1.34123 \times 10^{-5} \end{bmatrix}$$

From Equations (89) and (90), it is also required that

$$\bar{Y}_{13} = [0] \tag{128}$$

and

$$\bar{Y}_{43} = \bar{Y}_{53} \tag{129}$$

All of the submatrices involved in Equation (91) are now specified in Equations (121), (123), (124), (125), (126), and (127). The remaining submatrices in Equation (76) must also be determined; remembering that $\bar{Y}^t_{ij} = \bar{Y}_{ji}$ for i = 1,2,3,4,5 and j = 1,2,3,4,5, those which remain are $\bar{Y}_{22}$, $\bar{Y}_{33}$, $\bar{Y}_{44}$, $\bar{Y}_{55}$, $\bar{Y}_{23}$, $\bar{Y}_{35}$, and $\bar{Y}_{45}$. The off-diagonal submatrices must be chosen so that each element is a negative RC driving-point admittance function and so that their poles are contained in the set of poles belonging to $\bar{Y}_{42}$ and $\bar{Y}_{52}$. The constants K_1 and K_2 are chosen so that row 1 satisfies the dominance condition, and the diagonal terms may be chosen as RC driving-point admittance functions with poles at $s = -4$, $s = -1.06567$, and $s = -0.93746$. A reduction in the number of elements may be obtained by choosing the diagonal terms so that rows 2, 3, 4, and 5 satisfy the dominance condition with equality.

The short-circuit admittance matrix of the 10-port 11-terminal transformerless grounded passive RC network has been determined, and the procedure given in reference (15) may be used for realization of the network after the admittance matrix has been expanded in its Foster form. After connection of four ideal operational amplifiers to ports 3 through 10, the short-circuit admittance matrix in Equation (120) is realized at ports 1 and 2.

CHAPTER IV

SIMULTANEOUS REALIZATION OF TWO ADMITTANCES WITH ONE OPERATIONAL AMPLIFIER

Occasionally, it may be desirable to realize a two-port network in a manner such that two of its four short-circuit admittance parameters are prescribed. For instance, a nonreciprocal two-port network with prescribed transmission zeros may be desired. Another instance when this type of synthesis procedure might be useful is in the realization of a prescribed open-circuit voltage transfer function for a two-port such as in Figure 4 where this function is given by

$$\frac{E_2}{E_1} = - \frac{Y_{21}}{Y_{22}} \tag{130}$$

In this case there is some freedom in the choice of Y_{21} and Y_{22}, but their ratio is fixed.

Also, it might be desirable to realize a voltage transfer ratio with a two-port terminated in a load admittance Y_L which is an arbitrary positive-real function of the complex-frequency variable. The voltage ratio of the network in Figure 5 is given by

$$\frac{E_2}{E_1} = - \frac{Y_{21}}{Y_{22} + Y_L} \tag{131}$$

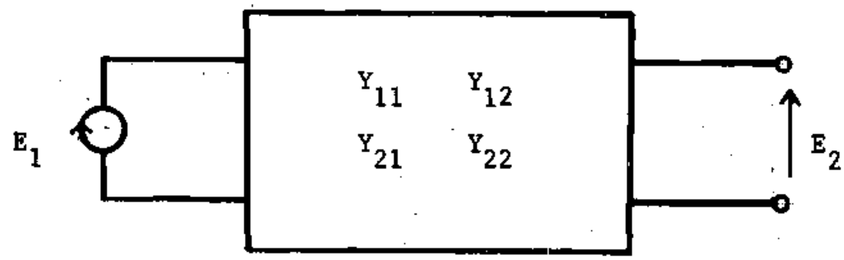


Figure 4. Two-Port Network Driven By Voltage Source.

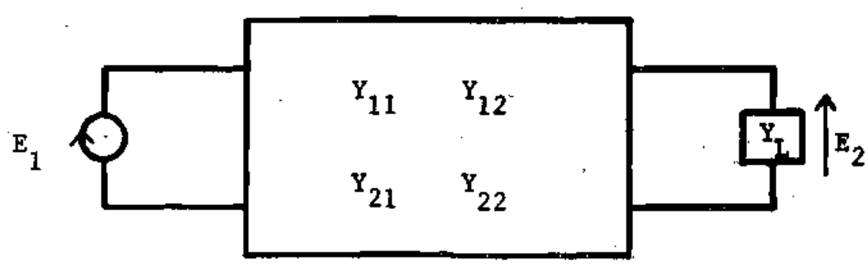


Figure 5. Two-Port Network Driven By Voltage Source and Terminated In An Admittance Y_L.

Realization of this function with a terminated two-port requires that Y_{21} and Y_{22} be specified independently.

Using the matrix synthesis procedures which have already been developed, it is possible to use two operational amplifiers and an RC network to realize all four short-circuit admittances simultaneously. This would be a solution to the problem, but it appears to be rather wasteful since the network is required to meet two additional unnecessary conditions. If one of the operational amplifiers could be eliminated from the network, a considerable saving in elements would result since two of the six ports with their associated elements would be eliminated in addition to the operational amplifier. The following theorem will now be proved:

Theorem 3

To realize two arbitrary short-circuit admittance parameters, which are real rational functions in the complex-frequency variable, with a 2-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the network contains one ideal operational amplifier.

The proof of this theorem can be accomplished by demonstrating a realization procedure for any pair of short-circuit admittances which are real rational functions of the complex-frequency variable. This proof differs from previous ones in that six realization procedures, one for each pair of parameters, must be developed, but the basic approach will remain the same. The short-circuit admittance parameters of the active network will first be found in terms of the admittance parameters of the passive portion of the network. The two prescribed parameters will then be compared with the two corresponding equations so as to allow identification of a realizable set of admittance parameters for the passive network.

Consider the network in Figure 2 and let N = 2. From Equation (40), the short-circuit admittance matrix of the active network for this special case is given by

$$[Y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} - \frac{1}{y_{34}} \begin{bmatrix} y_{13}\, y_{14} & y_{13}\, y_{24} \\ y_{23}\, y_{14} & y_{23}\, y_{24} \end{bmatrix} \tag{132}$$

where

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [Y] \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{133}$$

Employ the notation

$$[Y] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \tag{134}$$

so that

$$Y_{11} = y_{11} - \frac{y_{13}\, y_{14}}{y_{34}} \tag{135}$$

$$Y_{12} = y_{12} - \frac{y_{13}\, y_{24}}{y_{34}} \tag{136}$$

$$Y_{21} = y_{12} - \frac{y_{23}\, y_{14}}{y_{34}} \tag{137}$$

and

$$Y_{22} = y_{22} - \frac{y_{23}\, y_{24}}{y_{34}} \tag{138}$$

Now that analysis of the active network has been accomplished, the realization problem can be considered. Since there are four parameters which may be prescribed two at a time, there are six possible pairs which may be realized simultaneously. Each case will be considered separately, so that six realization procedures will be developed.

Case 1: Y_{11} and Y_{22}

Let the two prescribed admittance parameters be denoted by

$$Y_{11} = \frac{P_{11}}{Q} \tag{139}$$

and

$$Y_{22} = \frac{P_{22}}{Q} \tag{140}$$

where P_{11}, P_{22}, and Q are polynomials in the complex-frequency variable and Q is the least common denominator for Y_{11} and Y_{22}. Choose two RC driving-point admittance functions

$$y_{11} = \frac{p_{11}}{q} \tag{141}$$

and

$$y_{22} = \frac{p_{22}}{q} \tag{142}$$

where deg q > max (deg P_{11}, deg P_{22}, deg Q) and both y_{11} and y_{22} are regular at infinity. Subtracting Equation (141) from Equation (139) and Equation (142) from Equation (140) yields

$$Y_{11} - y_{11} = \frac{q\ P_{11} - Q\ p_{11}}{Q\ q} = \frac{B_1}{Q\ q} \tag{143}$$

and

$$Y_{22} - y_{22} = \frac{q\ P_{22} - Q\ p_{22}}{Q\ q} = \frac{B_2}{Q\ q} \tag{144}$$

where the polynomials B_1 and B_2 are defined as indicated. Equations (143) and (144) yield the following information regarding the degrees of the polynomials B_1 and B_2:

$$\deg B_1 \leq \max\ [(\deg q + \deg P_{11}),\ (\deg Q + \deg p_{11})] < 2 \deg q \tag{145}$$

and

$$\deg B_2 \leq \max\left[(\deg q + \deg P_{22}), (\deg Q + \deg p_{22})\right] < 2 \deg q \tag{146}$$

Therefore, the polynomials B_1 and B_2 can be factored so that

$$B_1 = -D_1 D_2 \tag{147}$$

and

$$B_2 = -D_3 D_4 \tag{148}$$

where the degree of each real polynomial factor is less than or equal to the degree of q.

Equations (135) and (138) can now be written as

$$\frac{y_{13} y_{14}}{y_{34}} = \frac{D_1 D_2}{q Q} \tag{149}$$

and

$$\frac{y_{23} y_{24}}{y_{34}} = \frac{D_3 D_4}{q Q} \tag{150}$$

Constants K_1 and K_2, to be specified later, may be introduced into Equations (149) and (150) and the right-hand sides rearranged so that the following identification can be accomplished

$$y_{13} = K_1 \frac{D_1}{q} \tag{151}$$

$$y_{14} = K_2 \frac{D_2}{q} \tag{152}$$

$$y_{23} = K_1 \frac{D_3}{q} \tag{153}$$

$$y_{24} = K_2 \frac{D_4}{q} \tag{154}$$

and

$$y_{34} = K_1 K_2 \frac{Q}{q} \tag{155}$$

Note that each parameter is regular at infinity because of the manner in which the degrees of the D's and q are specified.

The admittance parameters of the passive network have now been chosen so as to satisfy Equations (135) and (138). Any parameters not involved in these two equations may be chosen freely to facilitate the realization of the admittance matrix. If the short-circuit admittance matrix [y] is to be realized as a 4-port transformerless passive RC network, a set of sufficient conditions is as follows:

(1) the diagonal terms of [y] must be RC driving-point admittance functions, and

(2) if [y] is expressed in its Foster expansion, the coefficient matrices at each pole must be dominant.

Condition (1) is easily satisfied since y_{11} and y_{22} have been chosen to be RC driving-point admittance functions and y_{33} and y_{44}, which are arbitrary, may be chosen as such. Each of the off-diagonal parameters in Equations (151) through (155) has an unspecified constant multiplier which may be chosen as small as necessary to insure satisfaction of the dominance requirement in Condition (2). The only off-diagonal parameter which has not been specified is y_{12}, and it may be set equal to zero in order to decrease the complexity of the network. The two unspecified diagonal elements, y_{33} and y_{44}, may be chosen so as to satisfy the dominance requirements of rows three and four with equality and thus eliminate several elements.

The matrix [y] can now be realized using the technique developed by Weinberg and Slepian[14]. When an ideal operational amplifier is connected from port 4 to port 3 of the passive network, the desired driving-point admittance functions are realized at ports 1 and 2.

Case 2: Y_{12} and Y_{21}

Denote the two prescribed short-circuit admittance parameters Y_{12} and Y_{21} by

$$Y_{12} = \frac{P_{12}}{Q} \tag{156}$$

and

$$Y_{21} = \frac{P_{21}}{Q} \tag{157}$$

where Q is the least common denominator of Y_{12} and Y_{21}. Choose an RC transfer admittance function y_{12} and denote it by

$$y_{12} = K_a \frac{P_{12}}{q} \tag{158}$$

where deg q > max (deg P_{12}, deg P_{21}, deg Q), y_{12} is regular at infinity, and K_a is a specified constant multiplier. If Equation (158) is subtracted from Equations (156) and (157),

$$Y_{12} - y_{12} = \frac{q\ P_{12} - K_a\ Q\ P_{12}}{q\ Q} = \frac{B_1}{q\ Q} \tag{159}$$

and

$$Y_{21} - y_{12} = \frac{q\ P_{21} - K_a\ Q\ P_{12}}{q\ Q} = \frac{B_2}{q\ Q} \tag{160}$$

As in Equations (147) and (148) of Case 1, B_1 and B_2 can be factored so that the degree of each factor is less than or equal to the degree of q.

Therefore, Equations (136) and (137) become

$$\frac{y_{13}\ y_{24}}{y_{34}} = \frac{D_1\ D_2}{q\ Q} \tag{161}$$

and

$$\frac{y_{23}\ y_{14}}{y_{34}} = \frac{D_3\ D_4}{q\ Q} \tag{162}$$

After rearranging the above two equations and introducing the constants

K_1 and K_2 which will be specified later, the following identification can be made:

$$y_{13} = K_1 \frac{D_1}{q} \tag{163}$$

$$y_{24} = K_2 \frac{D_2}{q} \tag{164}$$

$$y_{23} = K_1 \frac{D_3}{q} \tag{165}$$

$$y_{14} = K_2 \frac{D_4}{q} \tag{166}$$

and

$$y_{34} = K_1 K_2 \frac{Q}{q} \tag{167}$$

where each parameter is regular at infinity. None of the four driving-point admittance functions are involved in Equations (136) and (137). Therefore, y_{11}, y_{22}, y_{33}, and y_{44} are arbitrary, and they may be chosen so that [y] satisfies the two realizability conditions listed in Case 1. The two constants K_1 and K_2 may be assigned any convenient value, and the matrix can be made to satisfy the realizability conditions by choosing the four driving-point admittances correctly. A considerable savings in elements is possible if the matrix satisfies the dominance requirement with equality.[14]

The matrix may be realized using the same procedure as Case 1. Termination of ports 3 and 4 with an ideal operational amplifier results in the realization of the two prescribed transfer admittances at ports 1 and 2.

Case 3: Y_{11} and Y_{12}

Let the two prescribed admittance parameters be denoted by

$$Y_{11} = \frac{P_{11}}{Q} \tag{168}$$

and

$$Y_{12} = \frac{P_{12}}{Q} \tag{169}$$

where Q is the least-common denominator of Y_{11} and Y_{12}. From Equations (135) and (136) it can be seen that this case is obviously more difficult than the preceding two because the second terms on the right-hand sides contain two common terms rather than one as before.

As the first step, choose an RC driving-point admittance

$$y_{11} = K_a \frac{p_{11}}{q} \tag{170}$$

so that the following conditions are satisfied:

(A) K_a is a specified constant multiplier;

(B) $m = \deg q > \max(\deg P_{11}, \deg P_{12}, \deg Q)$;

(C) y_{11} is regular at infinity so that $\deg p_{11} = m$; and

(D) the zeros of p_{11} and q, which must be on the negative-real axis, are placed in a region in which neither Q nor P_{11} changes sign.

Subtracting Equation (170) from Equation (168) gives

$$Y_{11} - y_{11} = \frac{q\ P_{11} - K_a\ Q p_{11}}{q\ Q} = \frac{B}{Q\ q} \tag{171}$$

Condition (D) is imposed upon the choice of y_{11} for the purpose of forcing the polynomial B to have (M-1) simple real zeros; proof of this fact follows. Since p_{11} and q are the zeros and poles, respectively, of an RC driving-point function, the two polynomials must have distinct and alternating zeros. Therefore, p_{11} and q must assume the same value somewhere between any two adjacent zeros. According to Condition (D), the zeros of p_{11} and q are placed in a region on the negative-real axis where neither P_{11} nor Q changes sign. From this it follows that Qp_{11} and qP_{11} must still assume the same value somewhere between any pair of adjacent zeros. Thus, $B = q\ P_{11} - K_a\ Q\ p_{11}$ will have at least (m-1) zeros on the negative-real axis no matter what value is assigned to K_a. As K_a approaches infinity, the negative-real zeros of interest become those of p_{11}.

The polynomial B may now be factored so that

$$B = -D_1\ D_2 \tag{172}$$

where D_1 has (m-1) simple real zeros. It was shown in Equation (145) of Case 1 that B is of degree less than 2m. Thus, if D_1 is of degree (m-1), D_2 must be of degree less than or equal to m. Equations (135), (171), and (172) yield

$$\frac{y_{13}\, y_{14}}{y_{34}} = \frac{D_1\, D_2}{q\, Q} \tag{173}$$

After introduction of the constants K_1 and K_2, which will be specified later, the following identification can be made:

$$y_{13} = K_1 \frac{D_1}{q} \tag{174}$$

$$y_{14} = K_2 \frac{D_2}{q} \tag{175}$$

and

$$y_{34} = K_1\, K_2 \frac{Q}{q} \tag{176}$$

Let the two parameters y_{24} and y_{12} be denoted by

$$y_{24} = \frac{N_{24}}{q} \tag{177}$$

and

$$y_{12} = \frac{N_{12}}{q} \tag{178}$$

Substitution of Equations (169), (174), (176), (177), and (178) into Equation (136) yields

$$q\, P_{12} = Q\, N_{12} - \frac{D_1\, N_{24}}{K_2} \tag{179}$$

Solving the above equation for N_{24} yields

$$N_{24} = \frac{K_2 \, (Q \, N_{12} - q \, P_{12})}{D_1} \tag{180}$$

The polynomials Q, P_{12}, q, and D_1 are either prescribed or have been determined, so that N_{12} and N_{24} are the only unknown polynomials in Equation (180). These two polynomials must be determined so that Equation (180) is satisfied and so that the short-circuit admittance matrix [y] can be made dominant. If [y] is to be dominant, the degrees of N_{12} and N_{24} must be less than or equal to m. Since N_{24} must be a polynomial, $(Q \, N_{12} - q \, P_{12})$ must contain all the zeros of D_1. If this is true, N_{24} is of degree less than or equal to m since the denominator of Equation (180) is of degree (m-1) and the degree of the numerator is less than or equal to (2m-1).

The problem is now to determine N_{12} so that $(Q \, N_{12} - q \, P_{12})$ contains all the zeros of D_1. This can be accomplished by choosing N_{12} to be an (m-2) degree polynomial with undetermined coefficients and determining the (m-1) coefficients so that $(Q \, N_{12} - q \, P_{12})$ contains all (m-1) zeros of D_1. Let

$$N_{12} = d_{m-2} \, s^{m-2} + d_{m-3} \, s^{m-3} + \cdots + d_1 \, s + d_0 \tag{181}$$

and

$$D_1 = (s-z_1)(s-z_2) \cdots (s-z_{m-1}) \tag{182}$$

where the z's are real and distinct and the d's are undetermined coefficients. The required cancellation will occur only if

$$(Q\,N_{12} - q\,P_{12})\Big|_{s=z_i} = 0 \tag{183}$$

where $i = 1,2,3,\ldots,(m-1)$. Rearrangement of Equation (183) gives

$$N_{12}(z_i) = \frac{P_{12}(z_i)\,q(z_i)}{Q(z_i)} \tag{184}$$

where $i = 1,2,\ldots,(M-1)$. Substituting Equation (181) into the above equation yields

$$\begin{aligned} d_{m-2}\,z_i^{m-2} + d_{m-3}\,z_i^{m-3} + \cdots + d_1\,z_i + d_0 &= \frac{P_{12}(z_i)\,q(z_i)}{Q\,(z_i)} \\ &= f(z_i) \end{aligned} \tag{185}$$

where $i = 1,2,\cdots,(m-1)$ and the function f is defined as shown. Solving the above set of (m-1) linear equations using Cramer's rule yields

$$d_j = \frac{W_j}{W} \tag{186}$$

where

$$W_j = \begin{bmatrix} z_1^{m-2} & z_1^{m-3} & \cdots & z_1^{m-j+1} & f(z_1) & z_1^{m-j-1} & \cdots & z_1 & 1 \\ z_2^{m-2} & z_2^{m-3} & \cdots & z_2^{m-j+1} & f(z_2) & z_2^{m-j-1} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ z_{m-1}^{m-2} & z_{m-1}^{m-3} & \cdots & z_{m-1}^{m-j+1} & f(z_{m-1}) & z^{m-j-1} & \cdots & z_{m-1} & 1 \end{bmatrix} \quad (187)$$

$$W = \begin{bmatrix} z_1^{m-2} & z_1^{m-3} & \cdots & z_1 & 1 \\ z_2^{m-2} & z_2^{m-3} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_{m-1}^{m-2} & z_{m-1}^{m-3} & \cdots & z_{m-1} & 1 \end{bmatrix} \quad (188)$$

and $j = 1,2,\ldots,(m-1)$.

Note that W is also given by

$$W = \prod_{u>v} (z_u - z_v) \quad (189)$$

where $u = 1,2,\ldots,m-1$ and $v = 1,2,\ldots,m-1$.[12] Therefore, since D_1 has only simple zeros, the z's are distinct and $W \neq 0$. This implies that the d's in Equation (186) always exist, and the polynomial N_{12} may always be determined. After N_{12} is found, the polynomial N_{24} can be determined from equation (180).

All of the parameters involved in Equations (135) and (136) have now been determined so as to satisfy these equations. Those parameters of [y] not involved in the two equations may be chosen freely to facilitate realization of the short-circuit admittance matrix. If [y] is to be realizable as a passive RC network without transformers, it is sufficient that Conditions (1) and (2) of Case 1 be satisfied. Condition (1) is satisfied since y_{11} is initially chosen to be an RC driving-point admittance; and y_{22}, y_{33} and y_{44} are arbitrary and can be chosen to satisfy the condition.

All of the off-diagonal parameters except y_{12} have a constant multiplier which is as yet unspecified, and thus, they may be made arbitrarily small to aid in the satisfaction of Condition (2). Because of the magnitude of y_{12}, row one of [y] may not be dominant, in which case it is necessary to increase K_a in Equation (170), but this will also effect N_{12}. As K_a is increased, the (m-1) zeros of D_1 approach the zeros of p_{11} as is easily seen in Equation (171). The d's, and thus N_{12}, can be computed when D_1 consists of (m-1) of the zeros of p_{11}. From this computation it is possible to determine how large K_a must be in order to insure dominance of [y] if D_1 consists of the chosen (m-1) zeros of p_{11}. By choosing K_a large enough, while remembering that it must be larger than the value determined above, the zeros of D_1 may be forced arbitrarily close to the chosen (m-1) zeros of p_{11}, and thus, N_{12} approaches arbitrarily close to its value determined above. Therefore, the matrix [y] can always be made to have dominant coefficient matrices by increasing K_a sufficiently and choosing K_1 and K_2 small.

As in Case 1, the short-circuit admittance matrix [y] can be realized using the technique developed by Weinberg and Slepian. Connection of an ideal operational amplifier from port 4 to port 3 of the passive transformerless RC network yields the desired active network.

Case 4: Y_{11} and Y_{21}

Denote the two prescribed admittances Y_{11} and Y_{21} by

$$Y_{11} = \frac{P_{11}}{Q} \tag{190}$$

and

$$Y_{21} = \frac{P_{21}}{Q} \tag{191}$$

where Q is the least-common denominator of the two admittances. An examination of Equations (135), (136), and (137) reveals that this case is similar to the previous one, and the results which have already been obtained can be utilized.

As in Case 3, choose an RC driving-point admittance

$$y_{11} = K_a \frac{P_{11}}{q} \tag{192}$$

which satisfies Conditions (A) through (D) where P_{12} is replaced by P_{21}. The steps of the realization procedure are now the same as Case 3, so that the following identification can be made:

$$y_{13} = K_2 \frac{D_2}{q} \tag{193}$$

$$y_{14} = K_1 \frac{D_1}{q} \tag{194}$$

and

$$y_{34} = K_1 \; K_2 \frac{Q}{q} \tag{195}$$

where

$$D_1 \; D_2 = q \; P_{11} - K_a \; Q \; p_{11} \tag{196}$$

and D_1 is of degree (m-1) and has only distinct real zeros. Employ the notation

$$y_{12} = \frac{N_{12}}{q} \tag{197}$$

and

$$y_{23} = \frac{N_{23}}{q} \tag{198}$$

and use Equation (137) to obtain

$$N_{23} = \frac{K_2 \; (Q \; N_{12} - q \; P_{21})}{D_1} \tag{199}$$

An examination of Equations (199) and (180) reveals that they are of the same form, and the only difference is that N_{24}, P_{12}, and K_1 are replaced by N_{23}, P_{21}, and K_2, respectively. Therefore, the remainder of the realization procedure is the same as Case 3, and the same arguments may be used to prove that [y] is realizable with a passive RC network without transformers. The passive network can be realized in the same manner as in the previous cases. Connection of an ideal operational amplifier from port 4 to port 3 yields the desired active network.

Case 5: Y_{22} and Y_{21}

Using steps similar to those employed in the previous cases, a realization procedure could be developed for this pair of short-circuit admittances, but an examination of Equations (135), (136), (137), and (138) reveals a much simpler solution. Notice that if in Equations (137) and (138) all subscripts 1 and 2 are changed to 2 and 1, respectively, Equations (135) and (138) are the same, as are Equations (136) and (137), provided it is remembered that $y_{12} = y_{21}$. This implies that the realization procedure of Case 3 can be used for Case 5 if the indicated changes in subscripts are made in all the variables used in the procedure.

Case 6: Y_{22} and Y_{12}

As in the previous case, a close examination of Equations (135), (136), (137), and (138) reveals that the realization procedure developed in Case 4 may be used for Case 6 if the proper changes in subscripts are made. If the subscripts in Equations (136) and (138) are changed so that all 1's are replaced by 2's and all 2's are replaced by 1's, Equations (136) and (137) are the same, as are Equations (135) and (138). Thus, if all the subscripts in the realization procedure of Case 4 are changed as indicated, the procedure can be used to realize the pair Y_{22} and Y_{12}.

Realization procedures for all six pairs of short-circuit admittance parameters have now been developed, and proof of Theorem 3 is complete. An example of Case 5 will be presented in Chapter VI.

CHAPTER V

SIMULTANEOUS REALIZATION OF N ADMITTANCES WITH ONE OPERATIONAL AMPLIFIER

As an extension to the results which have already been obtained, it might be useful to determine how many and which ones of the N^2 admittance parameters of an N-port active transformerless RC network containing one ideal operational amplifier can be realized simultaneously. The number is obviously limited since it has already been proved in Chapter II that N ideal operational amplifiers are necessary for the realization of all N^2 parameters provided the set of parameters satisfies Condition (b) of Theorem 1. Also, only certain sets of the parameters may be realized simultaneously since otherwise a set could be chosen which includes Y_{11}, Y_{22}, Y_{12}, and Y_{21} and satisfies Condition (b) of Theorem 1 for the 2 x 2 matrix consisting of these four parameters. This is of course impossible since it has been proved in Chapter II that two ideal operational amplifiers are needed to realize these four admittance parameters simultaneously.

The results which have been obtained regarding the above problem are stated as Theorem 4 with the proof following the statement of the theorem.

<u>Theorem 4</u>

To realize N arbitrary short-circuit admittance parameters with an N-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the RC network contains one ideal operational amplifier provided

(a) the admittances are real rational functions in the complex-frequency variable and

(b) one admittance parameter is prescribed from each column (row) of the short-circuit admittance matrix of the active N-port network.

Since this theorem is a sufficiency condition, the proof can be accomplished by demonstrating a realization procedure for simultaneous realization of any N admittance parameters. The type network which must be used is shown in Figure 2 with the short-circuit admittance matrix [Y] of this network given in Equation (40). If [Y] is denoted by

$$[Y] = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \tag{200}$$

then Equation (40) yields

$$Y_{ab} = y_{ab} - \frac{y_{a(N+1)}\, y_{b(N+2)}}{y_{(N+1)(N+2)}} \tag{201}$$

where $a = 1,2,\ldots,N$, $b = 1,2,\ldots,N$ and $y_{ab} = y_{ba}$.

Let the N prescribed parameters be denoted by

$$Y_{ab} = \frac{P_{ab}}{Q} \tag{202}$$

where a and b may be integers between 1 and N, and Q is the least common-denominator of all the admittances.

Any of the N admittance which are driving-point functions will be considered initially. The ith driving-point admittance is given by

$$Y_{ii} = y_{ii} - \frac{y_{i(N+1)}\, y_{i(N+2)}}{y_{(N+1)(N+2)}} \tag{203}$$

Choose an RC driving-point admittance and use the notation

$$y_{ii} = K'_{ii} \frac{p_{ii}}{q} \tag{204}$$

where

(A) K'_{ii} is a specified constant multiplier;

(B) $m = \deg q$ and m is greater than the degree of Q or the degree of any of the P's;

(C) y_{ii} is regular at infinity; and

(D) the zeros of p_{ii} and q are placed in a region in which neither Q nor any of the P's change sign.

Subtracting Equation (204) from Y_{ii} gives

$$Y_{ii} - y_{ii} = \frac{q\, P_{ii} - K'_{ii}\, Q\, p_{ii}}{Q\, q} = \frac{B_{ii}}{q\, Q} \tag{205}$$

As in Case 3 of the previous chapter, it can be proved that the polynomial B_{ii} contains (m-1) simple real zeros. Therefore

$$B_{ii} = -D_{1ii}\, D_{2ii} \tag{206}$$

where D_{2ii} is of degree less than or equal to m, and D_{1ii} is of degree (m-1) and has only simple real zeros. Substitution of Equations (205) and (206) into Equation (203) yields

$$\frac{y_{i(N+1)}\, y_{i(N+2)}}{y_{(N+1)(N+2)}} = \frac{D_{1ii}\, D_{2ii}}{Q\, q} \tag{207}$$

and identification of the three parameters can be made in the following manner:

$$y_{i(N+1)} = K_1 \frac{D_{1ii}}{q} \tag{208}$$

$$y_{i(N+2)} = K_2 \frac{D_{2ii}}{q} \tag{209}$$

and

$$y_{(N+1)N+2)} = K_1\, K_2 \frac{Q}{q} \tag{210}$$

where K_1 and K_2 are constants which will be specified later. Using this same procedure, the remaining prescribed driving-point admittance functions are considered in any order.

After determining all the y's involved in the equations for the prescribed driving-point functions, any pair of prescribed parameters Y_{uv} and Y_{vu} is considered. From Equation (201),

$$Y_{uv} = y_{uv} - \frac{y_{u(N+1)}\, y_{v(N+2)}}{y_{(N+1)(N+2)}} \tag{211}$$

and

$$Y_{vu} = y_{uv} - \frac{y_{v(N+1)}\, y_{u(N+2)}}{y_{(N+1)(N+2)}} \tag{212}$$

Rather than attempting to prove the theorem for both cases of Condition (b), it will be assumed that one admittance parameter is prescribed from each column of the short-circuit admittance matrix of the active N-port network. This implies that Y_{uu} and Y_{vv} can not be included in the set of prescribed functions. If it is noted that only driving-point admittances have been considered previously, Equation (40) reveals that $y_{u(N+1)}$, $y_{v(N+2)}$, $y_{v(N+1)}$, and $y_{u(N+2)}$ are at this point undetermined. Also, y_{uv} is still unspecified since it appears only in Y_{uv} and Y_{vu}. Thus, Y_{uv}, Y_{vu}, and $y_{(N+1)(N+2)}$ are the only known parameters in Equations (211) and (212). The technique in Case 2 of Chapter IV may be used for identification purposes provided $y_{u(N+1)}$ and $y_{v(N+1)}$ are determined so as to possess only distinct real zeros.

Choose an RC driving-point admittance and denote it by

$$y_{uv} = K'_{uv} \frac{p_{uv}}{q} \tag{213}$$

where

(A) K'_{uv} is a specified constant multiplier;

(B) y_{uv} is regular at infinity; and

(C) the zeros of p_{uv} and q are placed in a region in which neither Q nor any of the P's change sign.

Subtracting Equation (213) from Y_{uv} and Y_{vu} gives

$$Y_{uv} - y_{uv} = \frac{q\,P_{uv} - K'_{uv}\,Q\,p_{uv}}{Q\,q} = \frac{B^{(1)}_{uv}}{q\,Q} \tag{214}$$

and

$$Y_{vu} - y_{uv} = \frac{q\,P_{uv} - K'_{uv}\,Q\,p_{uv}}{Q\,q} = \frac{B^{(2)}_{uv}}{q\,Q} \tag{215}$$

Using a proof similar to that of Case 3, it is possible to prove that $B^{(1)}_{uv}$ and $B^{(2)}_{uv}$ can be factored so that

$$B^{(1)}_{uv} = -D^{(1)}_{1uv}\,D^{(1)}_{2uv} \tag{216}$$

and

$$B^{(2)}_{uv} = -D^{(2)}_{1uv}\,D^{(2)}_{2uv} \tag{217}$$

where $D^{(1)}_{1uv}$ and $D^{(2)}_{1uv}$ are each of degree (m-1) and contain only simple real zeros. Also, $D^{(1)}_{2uv}$ and $D^{(2)}_{2uv}$ are of degree less than or equal to m. The following equations can now be obtained by substituting Equations (214) through (217) into Equations (211) and (212):

$$\frac{y_{u(N+1)}\, y_{v(N+2)}}{y_{(N+1)(N+2)}} = \frac{D^{(1)}_{1uv}\, D^{(1)}_{2uv}}{q\, Q} \tag{218}$$

and

$$\frac{y_{v(N+1)}\, y_{u(N+2)}}{y_{(N+1)(N+2)}} = \frac{D^{(2)}_{1uv}\, D^{(2)}_{2uv}}{q\, Q} \tag{219}$$

After introducing the constants K_1 and K_2 the identification shown below can be made.

$$y_{u(N+1)} = K_1 \frac{D^{(1)}_{1uv}}{q} \tag{220}$$

$$y_{v(N+2)} = K_2 \frac{D^{(1)}_{2uv}}{q} \tag{221}$$

$$y_{v(N+1)} = K_1 \frac{D^{(2)}_{1uv}}{q} \tag{222}$$

and

$$y_{u(N+2)} = K_2 \frac{D^{(2)}_{2uv}}{q} \tag{223}$$

Similarly, all other such pairs of prescribed admittances are considered. No two pairs can involve the same row since this implies that they must also involve the same column which is impossible. Therefore, for each pair of Y's, the only y-parameter which has been previously prescribed is

$$y_{(N+1)(N+2)} = K_1 K_2 \frac{Q}{q} \tag{224}$$

and the above identification technique can be used for each pair.

After all driving-point functions and admittance pairs have been used, the remainder of the parameters can be considered in random order. Assume that Y_{rt} is a prescribed admittance which has not yet been considered. From Equation (201),

$$Y_{rt} = y_{rt} - \frac{y_{r(N+1)}\, y_{t(N+2)}}{y_{(N+1)(N+2)}} \tag{225}$$

Since only one admittance per column may be prescribed, Equations (40) and (200) reveal that $y_{t(N+2)}$ must be unspecified at this stage of the realization procedure. Another prescribed parameter from row r may have been considered previously, in which case $y_{r(N+1)}$ will already be specified. Since y_{rt} appears only in the equations for Y_{rt} and Y_{tr}, it must be unspecified; if Y_{tr} were also included in the set of prescribed parameters, the pair would have been considered previously.

Assume first that $y_{r(N+1)}$ has already been determined. A procedure similar to the one used in Case 3 of Chapter IV will be used to determine y_{rt} and $y_{t(N+2)}$. Note that the numerator of $y_{r(N+1)}$ must be of degree (m-1) and must contain only distinct real zeros. Let the following notation be employed:

$$y_{rt} = \frac{N_{rt}}{q}$$

$$y_{r(N+1)} = K_1 \frac{D_{rt}}{q} \tag{227}$$

and

$$y_{t(N+2)} = \frac{N_{t(N+2)}}{q} \tag{228}$$

Equation (225) can now be rewritten as

$$N_{t(N+2)} = K_2 \frac{Q\ N_{rt} - q\ P_{rt}}{D_{rt}} \tag{229}$$

where N_{rt} and $N_{t(N+2)}$ remain to be determined, and D_{rt} is of degree (m-1) and possesses only distinct real zeros. An examination of Equation (180) reveals that it is of the same form as Equation (229), and the procedure used to determine N_{12} and N_{24} in Case 3 can be used to determine $N_{t(N+2)}$ and N_{rt}.

If $y_{r(N+1)}$ has not been previously determined, a different procedure must be employed for determining y_{rt}, $y_{r(N+1)}$, and $y_{t(N+2)}$. As before, $y_{r(N+1)}$ must contain (m-1) simple real zeros. The same procedure used for the pair Y_{uv} and Y_{vu} can be employed at this point. The parameter y_{rt} must be chosen to meet the same conditions as y_{uv}; and $y_{r(N+1)}$ and $y_{t(N+2)}$ can be determined in the same manner as $y_{u(N+1)}$ and $y_{v(N+2)}$, respectively.

Each of the remaining prescribed parameters is now considered in the manner indicated above. Thus, all of the y-parameters involved in the equations for the N prescribed admittances have been identified so as to satisfy the equations. The remaining parameters may be chosen freely to facilitate realization of the matrix. A set of sufficient conditions for realizability of [y] as an (N+2)-port transformerless passive RC network is as follows:

(1) the diagonal terms are RC driving-point admittance functions and

(2) if the matrix is expanded in its Foster form, the coefficient matrices are all dominant.

Condition (1) is satisfied since all driving-point admittances which have been specified are required to meet the condition, and the remaining ones are arbitrary and can be chosen to satisfy it.

All off-diagonal parameters which are specified from consideration of prescribed driving-point functions as in Equations (208), (209), and (210), have an undetermined constant multiplier K_1, K_2, or K_1K_2. These parameters can be made as small as necessary merely by choosing K_1 and K_2 small. Next, examine the off-diagonal parameters which are specified from consideration of pairs of prescribed admittances such as Y_{uv} and Y_{vu}. Note that the diagonal terms on rows u and v of [y] must therefore be arbitrary since Y_{uu} and Y_{vv} can not be contained in the set of prescribed functions. Consequently, there is no need to consider any off-diagonal terms on rows u and v since the diagonal terms can be made as large as necessary to insure satisfaction of the dominance condition. This also implies that the only rows of the matrix [y] which present any trouble in regard to satisfying Condition (2) are those which correspond to the rows of [Y] from which driving-point admittance functions are prescribed. The reason for this is that these are the only rows in which the diagonal y-parameters are specified; all other diagonal terms are arbitrary and can be chosen as large as necessary.

Examine the admittances in Equations (227), (228), and (229) which are determined from consideration of Y_{rt}. Either the driving-point function Y_{rr} must be prescribed or one of a pair of prescribed functions must belong to row r of [Y]. If the driving-point function Y_{rr} is not prescribed, there is no problem since y_{rr} is arbitrary and can be chosen as large as necessary, but if Y_{rr} is prescribed, row r must be shown to satisfy the dominance condition. This can be accomplished by noting that $y_{r(N+1)}$ and $y_{r(N+2)}$ have undetermined constant multipliers and using the same argument on y_{rt} as was used on y_{12} in Case 3 of Chapter V. That is, it can be shown that K'_{rr} in Equation (204) can always be chosen large enough so that row r is dominant.

Both realizability conditions are thus satisfied, and the short-circuit admittance matrix [y] can be realized as a (N+2)-port transformerless balanced passive RC network. As in Chapter IV, the technique of Weinberg and Slepian will be used. If an ideal operational amplifier is connected from port (N+2) to port (N+1), the prescribed functions will be realized at ports 1 through N.

Proof of the theorem for the case when one admittance parameter is prescribed from each row of the short-circuit admittance matrix of the active N-port network is very similar to the case which has been presented, and it will not be given.

CHAPTER VI

EXPERIMENTAL RESULTS

In order to demonstrate that the realization procedures which have developed are not only correct but also practical, examples were worked using the procedures, and the resulting networks were constructed and tested. The test data obtained from the networks was then compared with the predicted behavior. An example network was constructed for each of the realization procedures in Chapters II and III and for Case 5 of Chapter IV.

Example 1

To illustrate realization procedure 1 of Chapter II, the simplest case of N = 1, that is a driving-point admittance, was chosen. Since an active RC network was to be used for the realization, the prescribed function was chosen to be a 10-henry inductor. The passive network resulting from the realization procedure had to be frequency and magnitude scaled to obtain realistic values for the elements. Consequently, using some foresight, the function $\bar{Y} = 1/s$ was realized initially. Frequency scaling of the network by a factor of 10^3 and magnitude scaling by 10^4 gave the desired driving-point admittance, $\bar{Y} = 1/10s$.

As the first step in the realization, $\bar{Y}_{11}$ was chosen as

$$\bar{Y}_{11} = \frac{s+1}{s+2} = \frac{[p]}{q} \tag{230}$$

Equation (22) gave

$$[B] = s^2 - 2 \tag{231}$$

so that

$$[D_1] = s - 1.414 \tag{232}$$

and

$$[D_2] = s + 1.414 \tag{233}$$

The parameter $\bar{Y}_{12}$ was then found from Equation (26).

$$\bar{Y}_{12} = K_1 \frac{s - 1.414}{s + 2} \tag{234}$$

Next, $\bar{Y}_{31}$ was chosen to be

$$\bar{Y}_{31} = K_2 \frac{1}{s + 2} = K_2 \frac{[G_{31}]}{q} \tag{235}$$

Using Equation (30),

$$\bar{Y}_{32} = K_1 K_2 \frac{s}{(s + 1.414)(s + 2)} \tag{236}$$

Rather than specifying $\bar{Y}_{22}$ and $\bar{Y}_{33}$ explicitly, they were allowed to assume the values necessary to make rows 2 and 3 of [y] satisfy the dominance condition with equality.

After choosing $K_1 = 0.22$ and $K_2 = 0.25$, Equations (230), (234), (235), and (236) were used to write

$$[y] = \begin{bmatrix} 0.5 & -0.1555 & 0.125 \\ -0.1555 & k_{22}^{(0)} & 0 \\ 0.125 & 0 & k_{33}^{(0)} \end{bmatrix} \tag{237}$$

$$+ \frac{s}{s + 2} \begin{bmatrix} 0.5 & 0.375 & -0.125 \\ 0.375 & k_{22}^{(1)} & -0.0938 \\ -0.125 & -0.0938 & k_{33}^{(1)} \end{bmatrix}$$

$$+ \frac{s}{s + 1.414} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{22}^{(2)} & 0.0938 \\ 0 & 0.0938 & k_{33}^{(2)} \end{bmatrix}$$

Networks realizing the three terms of [y] are shown in Figures 6, 7, and 8. These three networks were connected in parallel to realize [y]. Connection of an operational amplifier from port 3 to port 2 as shown in Figure 9 resulted in a transformerless active RC network which had the driving-point admittance $\bar{Y} = 1/s$ at port 1. Proper frequency and magnitude scaling of the network in Figure 9 gave the required function $\bar{Y} = 1/10s$ at port 1.

The network which was constructed had the capacitor and resistor values of Figures 6, 7, and 8 multiplied by 10^{-7} and 10^{4}, respectively. The magnitude and phase of the impedance at port 1 was measured. A comparison of the measured data with the predicted behavior is given in Figures 10 and 11.

Example 2

Realization procedure 2 of Chapter II was illustrated using the same prescribed function as in Example 1, that is, a 10-henry inductor. As was done in the previous example, the network realizing the function $[Y] = 1/s$ was synthesized, and the resulting network was frequency scaled by a factor of 10^{3} and magnitude scaled by 10^{4}. The prescribed function, $[Y] = 1/10s$, was then realized at the input port.

The function $[Y_{11}]$ was chosen as

$$[Y_{11}] = \frac{s + 1}{s + 2} = \frac{[p]}{q} \tag{238}$$

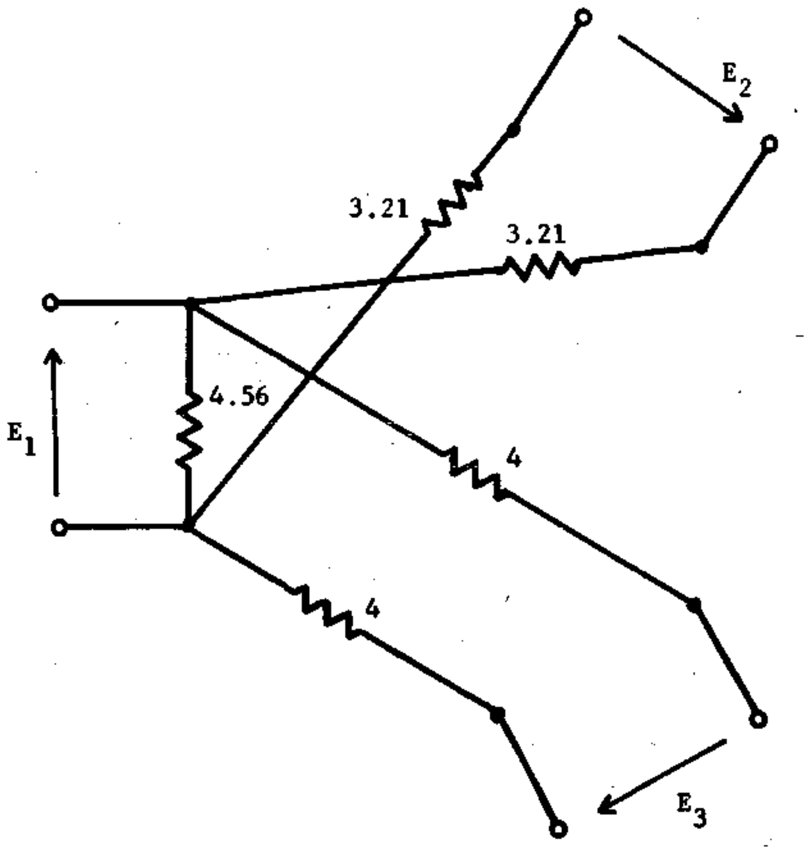
so that Equation (48) gave

$$[B] = s^2 - 2 \tag{239}$$

After factoring [B], the identification

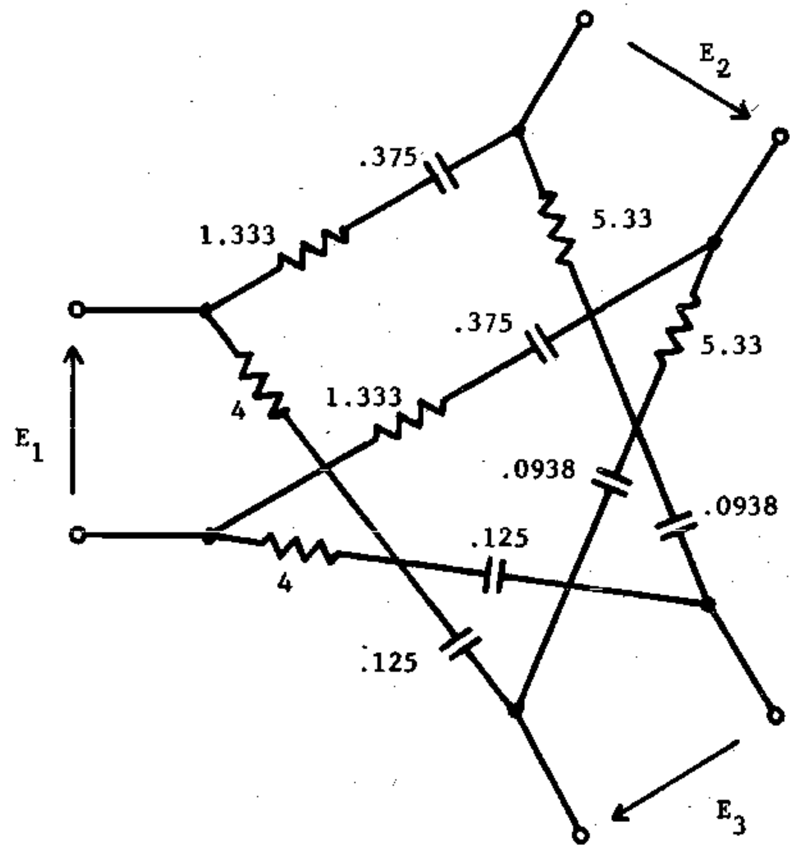
$$[D_1] = s - 1.414 = e_{11} \tag{240}$$

and



Ohms

Figure 6. Network Realizing the First Term of Equation (237).



Ohms, Farads

Figure 7. Network Realizing the Second Term of Equation (237).

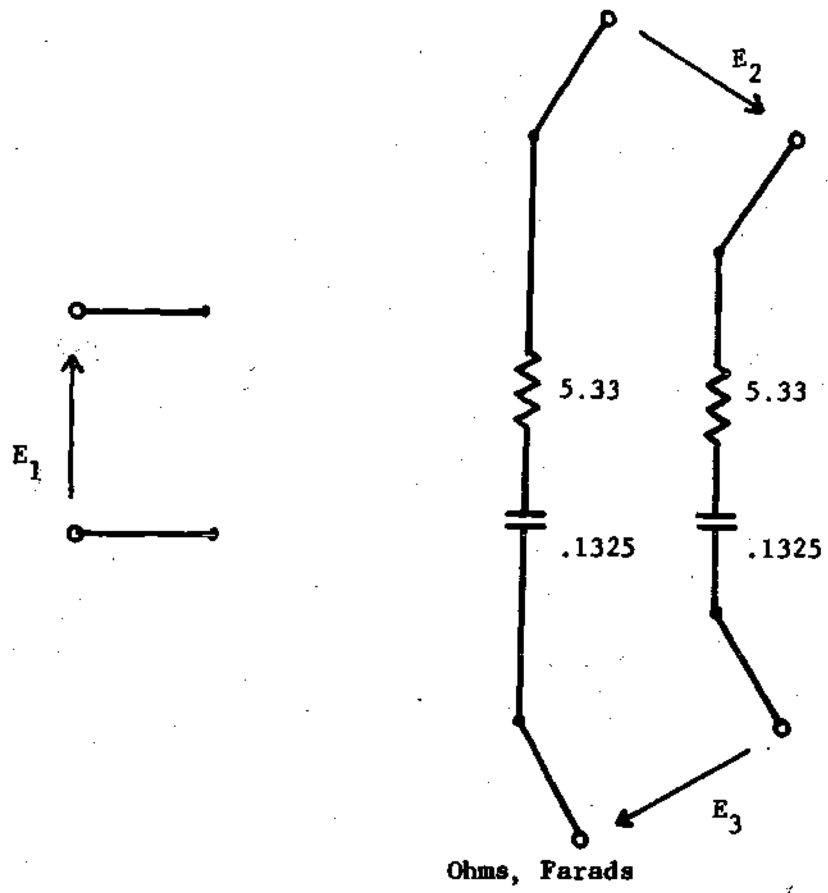


Figure 8. Network Realizing the Third Term of Equation (237).

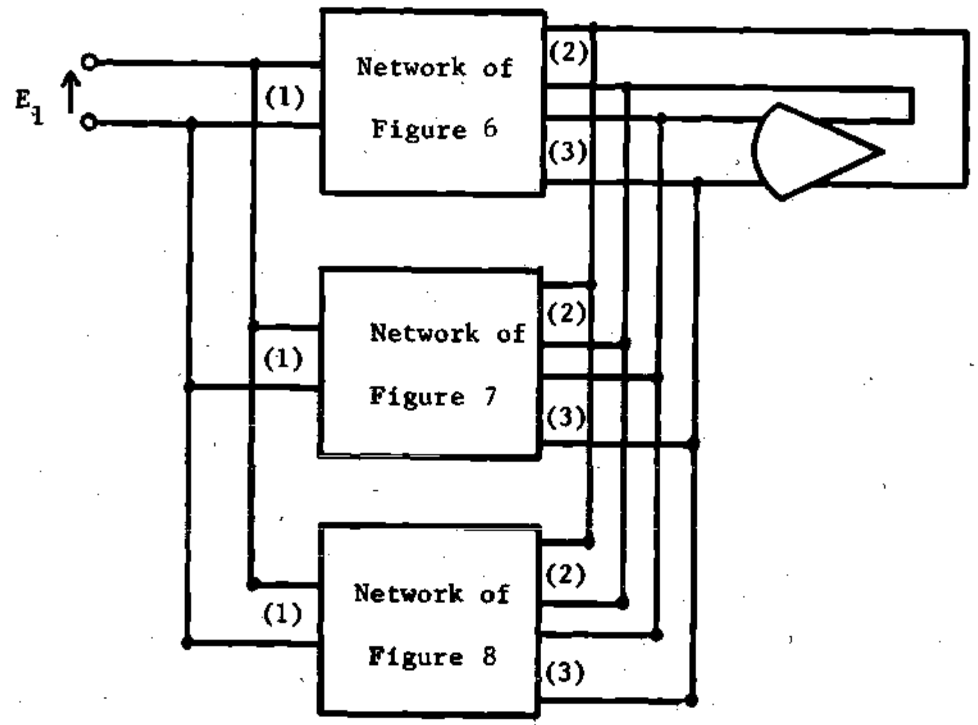


Figure 9. Network Realizing the Driving-Point Admittance $\overline{Y} = 1/s$.

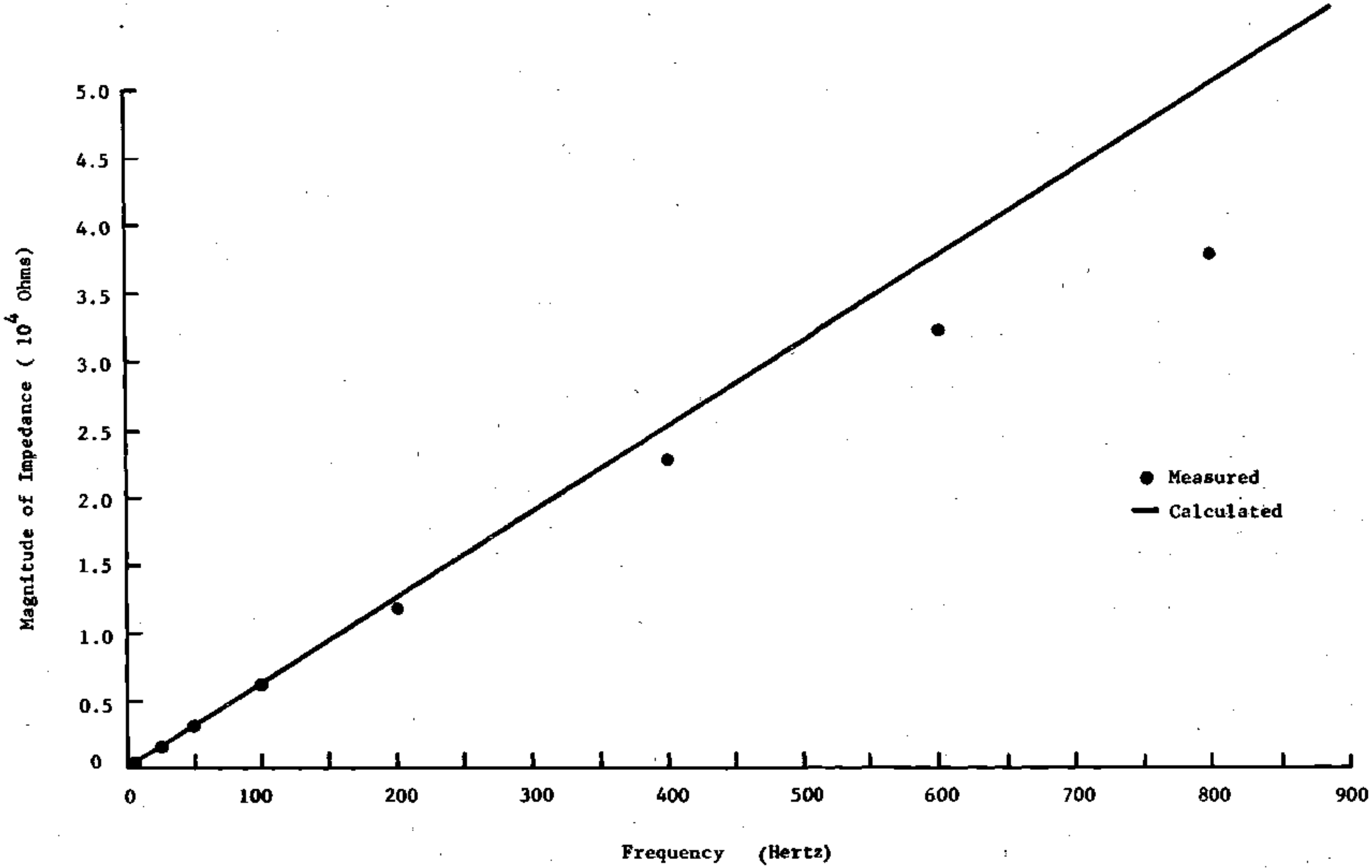


Figure 10. Comparison of Experimental Data with Desired Magnitude Variation for Ten Henry Inductor.

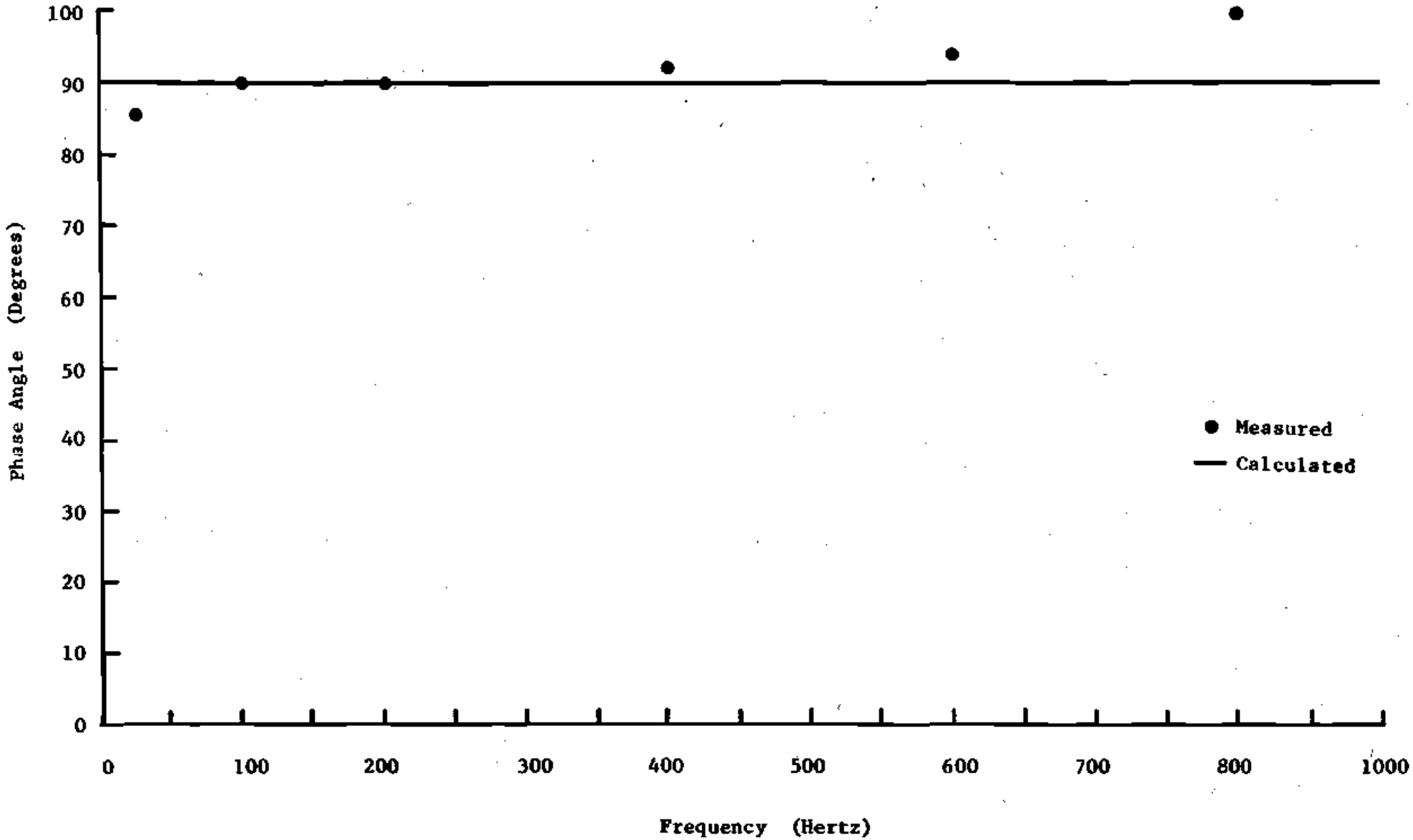


Figure 11. Comparison of Experimental Data with Desired Phase Variation for Ten Henry Inductor.

$$[D_2] = s + 1.414 = h_{11} \tag{241}$$

was made. Equations (56), (59), and (60) gave

$$[y_{12}^{(1)}] = K_1 \frac{s - 1.414}{s + 2} = K_1 \frac{e_{11}}{q} \tag{242}$$

$$[y_{13}^{(1)}] = K_2 \frac{s + 1.414}{s + 2} = K_2 \frac{h_{11}}{q} \tag{243}$$

and

$$[y_{23}^{(1)}] = K_1 \; K_2 \frac{s}{s + 2} = K_1 \; K_2 \frac{Q}{q} \tag{244}$$

The constants K_1 and K_2 were chosen to be

$$K_1 = 0.21 \tag{245}$$

and

$$K_2 = 0.501 \tag{246}$$

so that the short-circuit admittance matrix [y] for the passive network became

$$[y] = \begin{bmatrix} 0.5000 & -0.1483 & 0.3540 \\ -0.1483 & k_{22}^{(0)} & 0 \\ 0.3540 & 0 & k_{33}^{(0)} \end{bmatrix} \qquad (247)$$

$$+ \frac{s}{s+2} \begin{bmatrix} 0.5000 & 0.3585 & 0.1470 \\ 0.3585 & k_{22}^{(1)} & 0.1052 \\ 0.1470 & 0.1052 & k_{33}^{(1)} \end{bmatrix}$$

The parameters y_{22} and y_{33} were allowed to have the values necessary to make rows 2 and 3 of [y] satisfy the dominance requirement with equality.

The networks realizing the two terms of Equation (247) are shown in Figure 12 and 13, and the desired transformerless active RC network containing one ideal operational amplifier is shown in Figure 14. The network of Figure 14 realized the function $[Y] = 1/s$ as the driving-point admittance at port 1, but proper frequency and magnitude scaling of the RC networks yielded the prescribed admittance, $[Y] = 1/10s$, at port 1.

After scaling the network by multiplying capacitor values by 10^{-7} and resistor values by 10^{4}, it was constructed and tested. The magnitude and phase of the impedance which was measured at port 1 are compared with their predicted behavior in Figures 15 and 16.

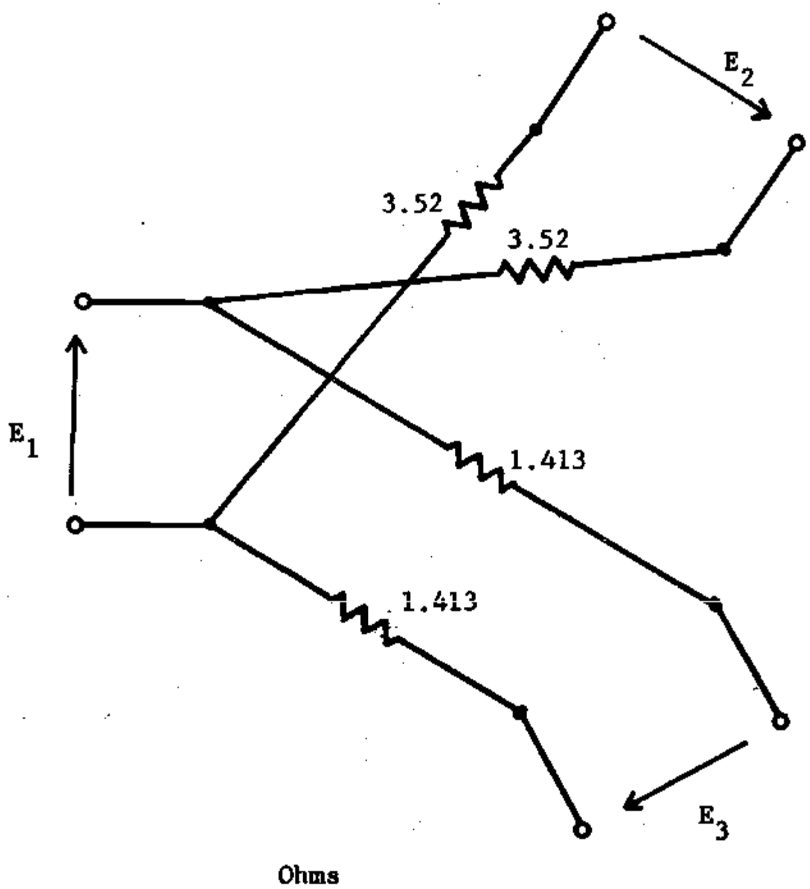


Figure 12. Network Realizing the First Term of Equation (247).

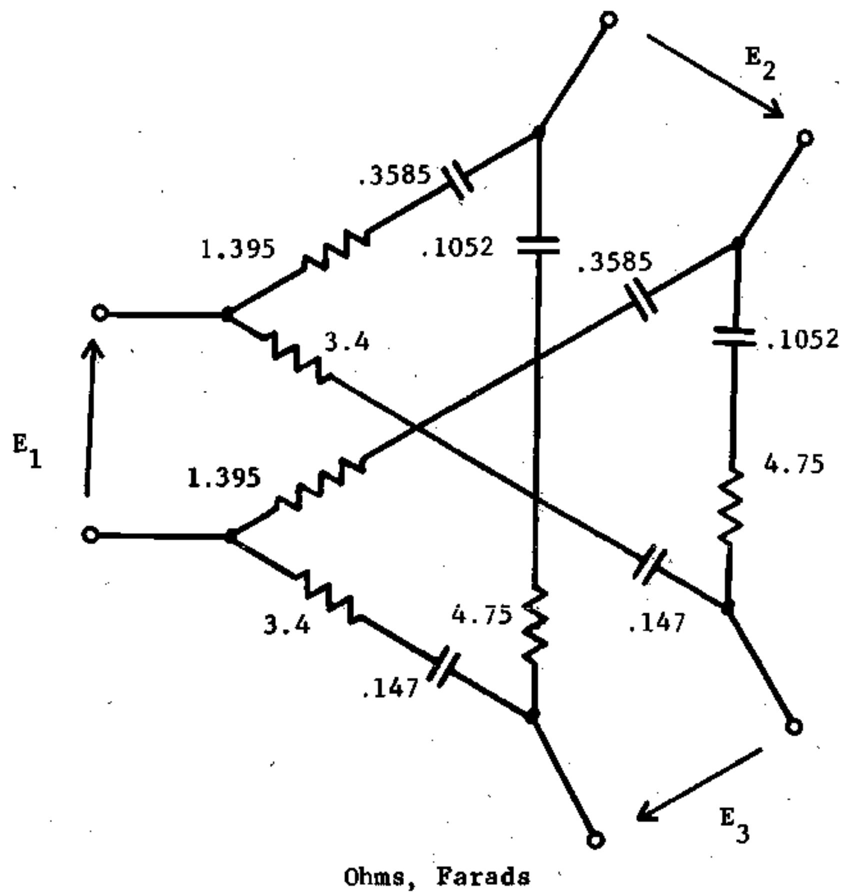


Figure 13. Network Realizing the Second Term of Equation (247).

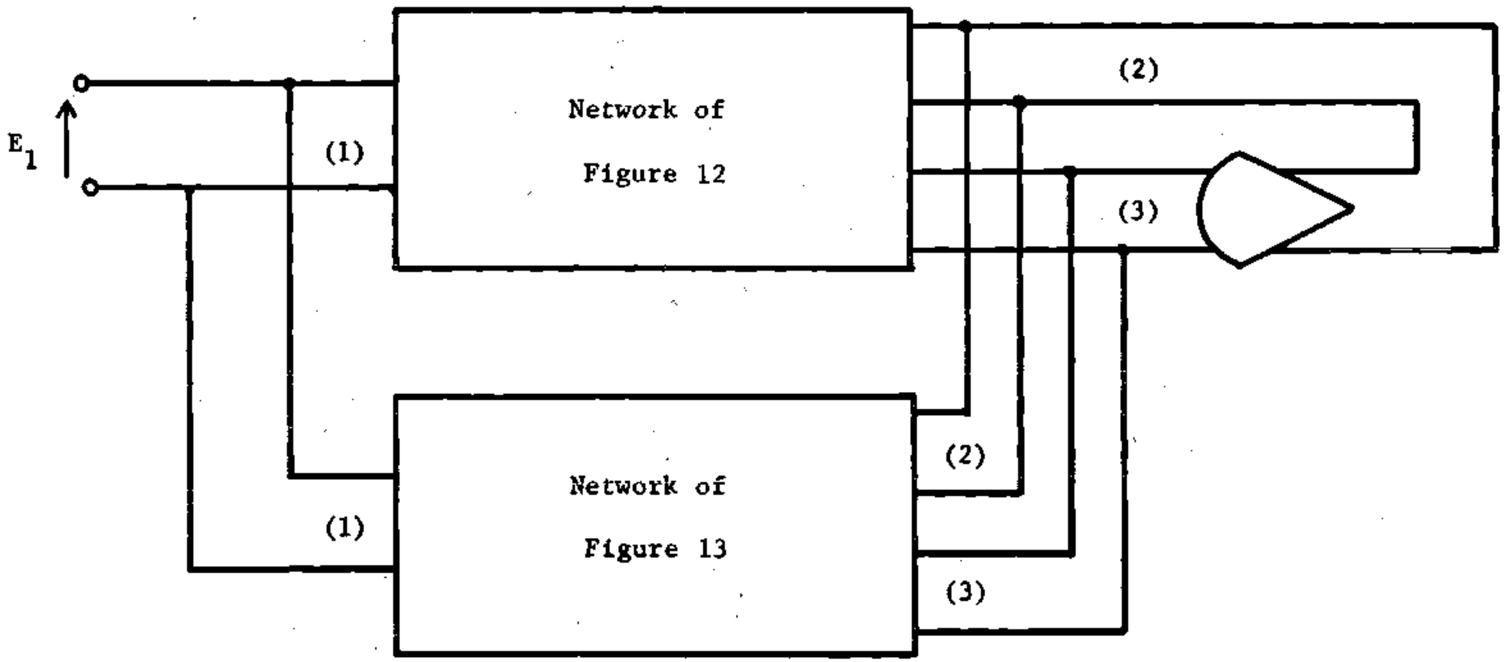


Figure 14. Network Realizing the Driving-Point Admittance [Y]= 1/s .

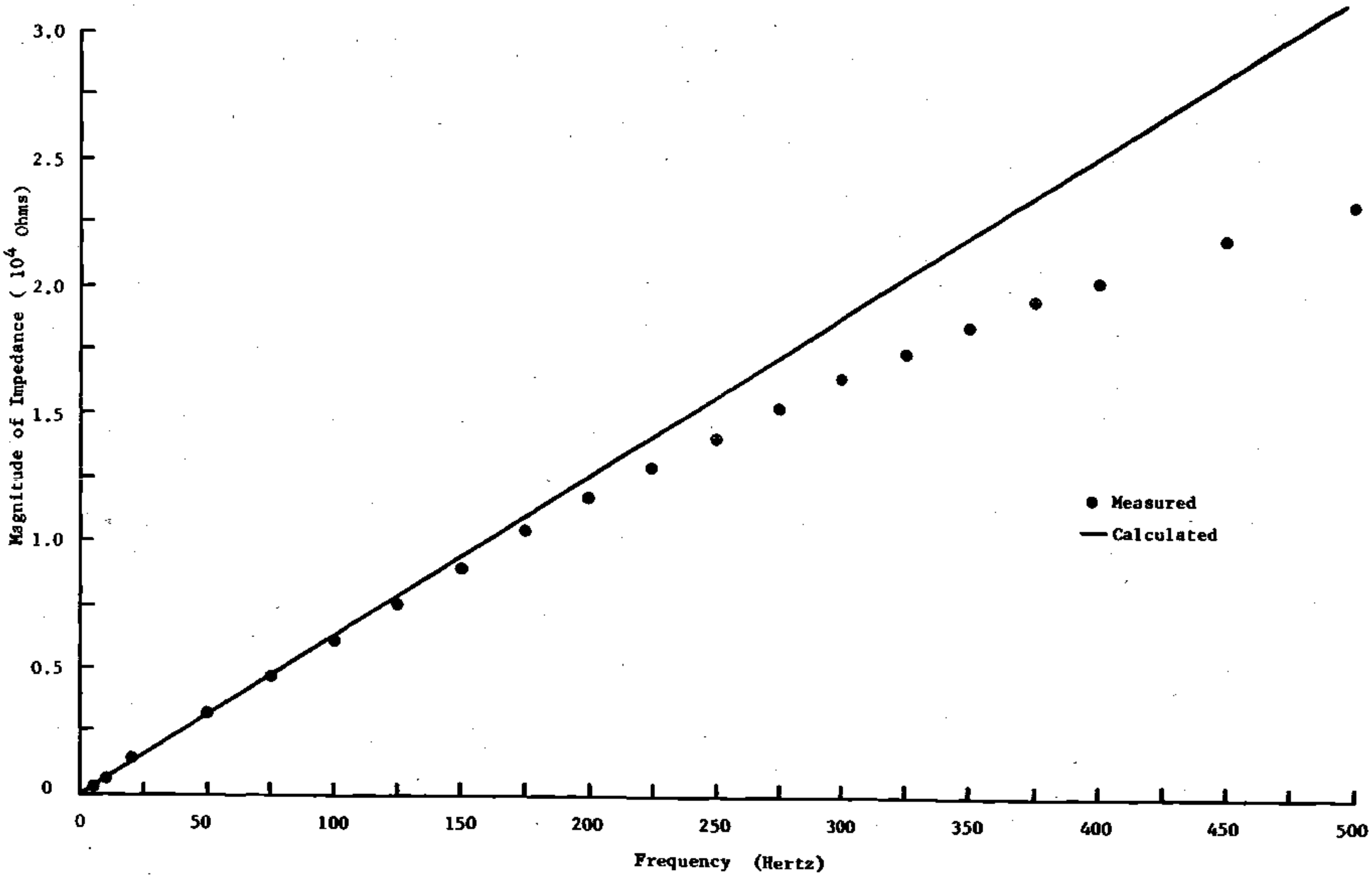


Figure 15. Comparison of Experimental Data with Desired Magnitude Variation for Ten Henry Inductor.

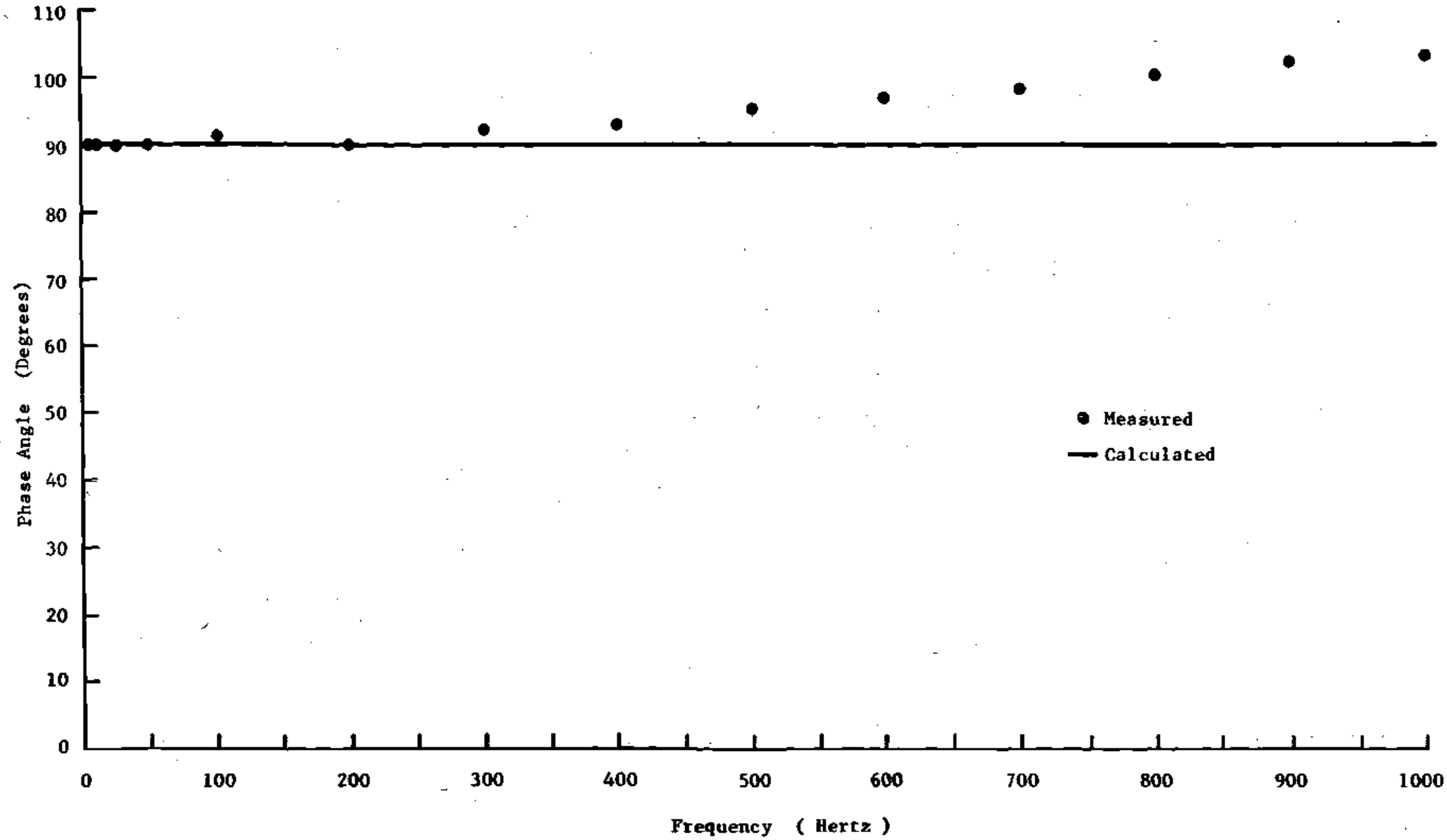


Figure 16. Comparison of Experimental Data with Desired Phase Angle Variation for Ten Henry Inductor.

Example 3

To illustrate the grounded realization procedure of Chapter III, a 1-port network was found having the prescribed driving-point admittance

$$\tilde{Y} = \frac{1}{s} \tag{248}$$

The admittance $\bar{Y}_{11}$ was chosen to be

$$\bar{Y}_{11} = \frac{s + 2}{s + 3} = \frac{[p]}{q} \tag{249}$$

so that Equation (94) gave

$$[B] = -s^2 - s + 3 \tag{250}$$

The function [B] was then factored, and $[D_b]$ and $[D_a]$ were identified as

$$[D_a] = -s + 1.305 \tag{251}$$

and

$$[D_b] = s + 2.305 \tag{252}$$

Using Equation (102),

$$\bar{Y}_{51} - \bar{Y}_{41} = K_1 \frac{-s + 1.305}{s + 3} \tag{253}$$

and the following identification was made:

$$\bar{Y}_{51} = -K_1 \frac{1.435s}{s + 3} \tag{254}$$

and

$$\bar{Y}_{41} = -0.435K_1 \tag{255}$$

The parameter $\bar{Y}_{12}$ was chosen to be

$$\bar{Y}_{12} = -K_2 \frac{s}{s + 3} \tag{256}$$

so that from Equation (105),

$$\bar{Y}_{42} - \bar{Y}_{52} = -K_1 K_2 \frac{s^2}{(s + 3)(s + 2.305)} \tag{257}$$

The parameters in the above equations were identified as

$$\bar{Y}_{42} = -K_1 K_2 \frac{4.32s}{s + 3} \tag{258}$$

and

$$\bar{Y}_{52} = -K_1 K_2 \frac{3.32s}{s + 2.305} \tag{259}$$

The short-circuit admittance matrix for the passive network was thus found to be

$$[y] = \begin{bmatrix} 0.670 & 0 & 0 & -0.087 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.000 & -0.500 & -0.500 \\ -0.087 & 0 & -0.500 & 0.587 & 0 \\ 0 & 0 & -0.500 & 0 & 0.500 \end{bmatrix} \tag{260}$$

$$+ \frac{s}{s+3} \begin{bmatrix} 0.3300 & -0.0439 & 0 & 0 & -0.2870 \\ -0.0430 & 0.0804 & 0 & -0.0374 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0374 & 0 & 0.0374 & 0 \\ -0.2870 & 0 & 0 & 0 & -0.2870 \end{bmatrix}$$

$$+ \frac{s}{s+2.305} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0285 & 0 & 0 & -0.0285 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0285 & 0 & 0 & 0.0285 \end{bmatrix}$$

where y_{22}, y_{33}, y_{44}, and y_{55} have been given the values necessary to make rows 2, 3, 4, and 5 of [y] satisfy the dominance condition with equality, and K_1 and K_2 have been chosen as 0.2 and 0.043, respectively.

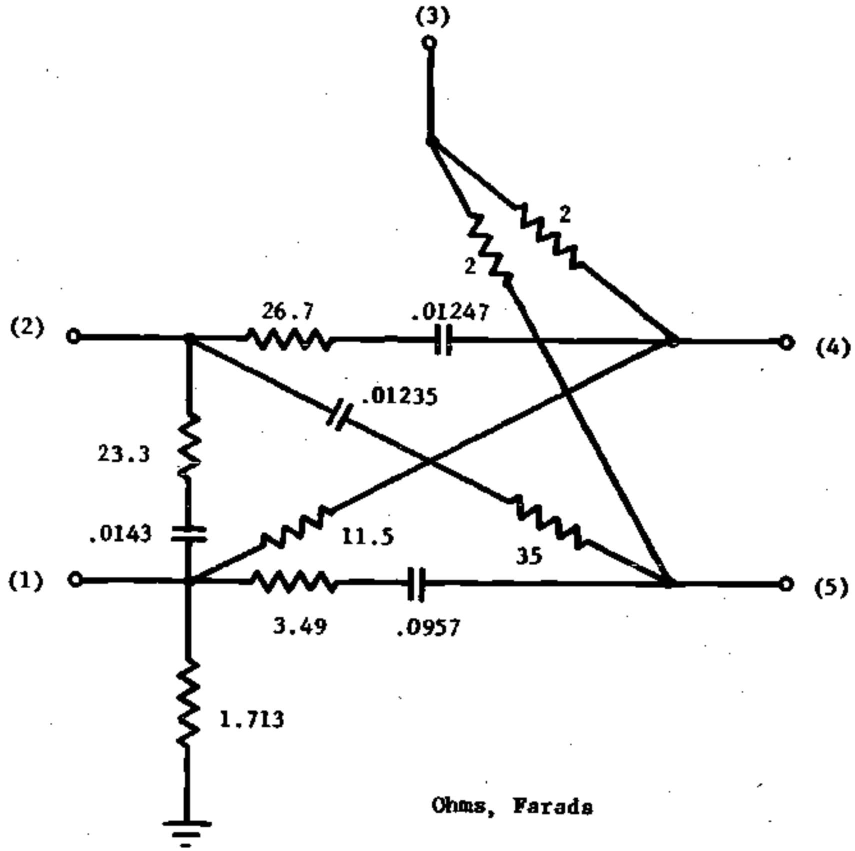


Figure 17. Network Realizing Equation (260).

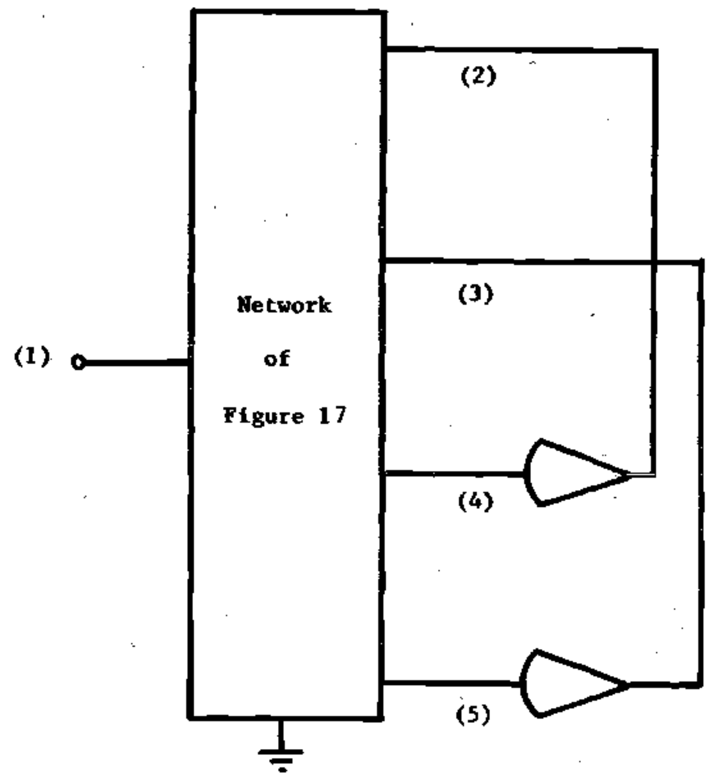


Figure 18. Network Realizing the Driving-Point Admittance $\overline{Y} = 1/s$.

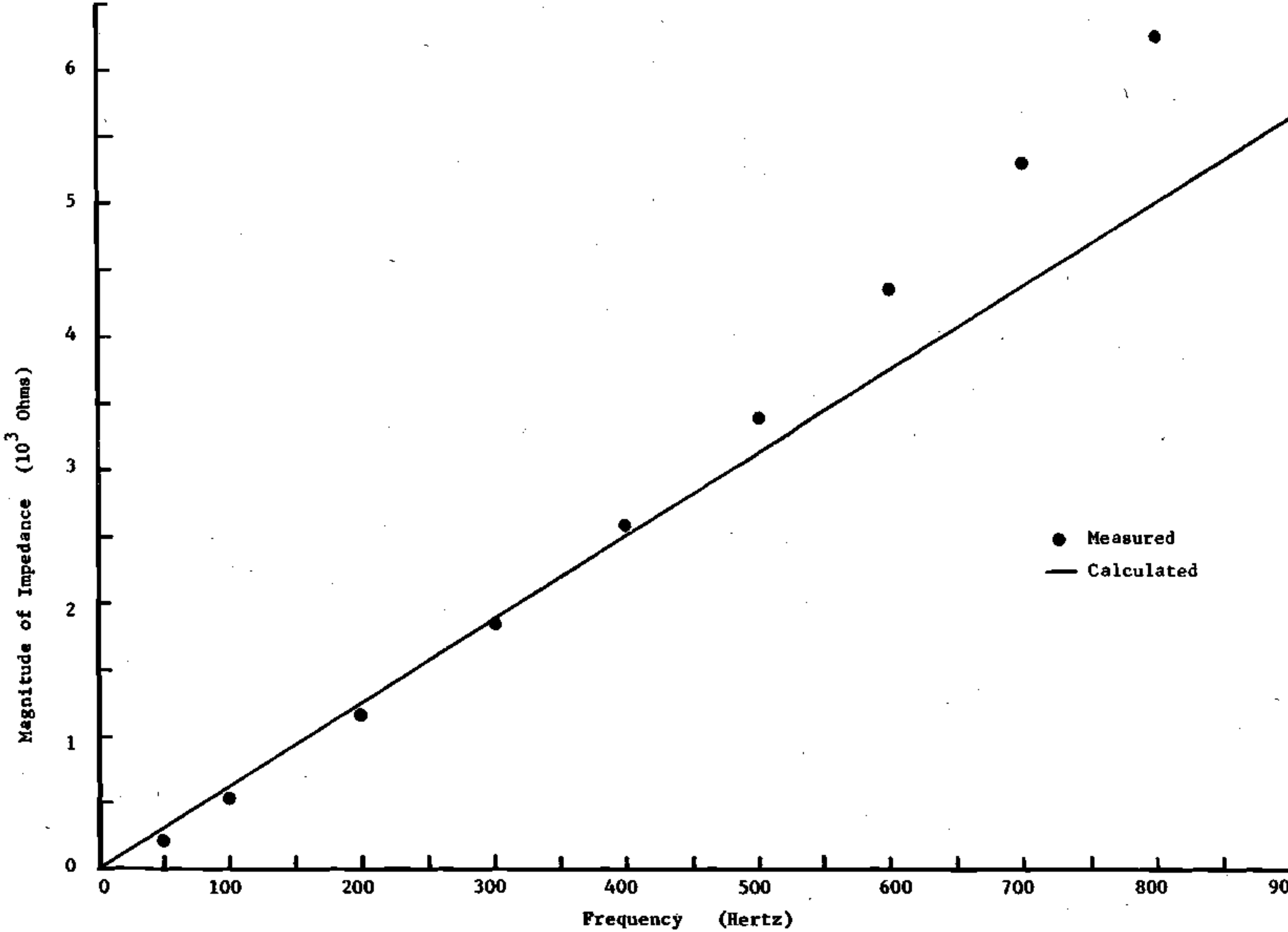


Figure 19. Comparison of Experimental Data with Desired Magnitude Variation for One Henry Inductor.

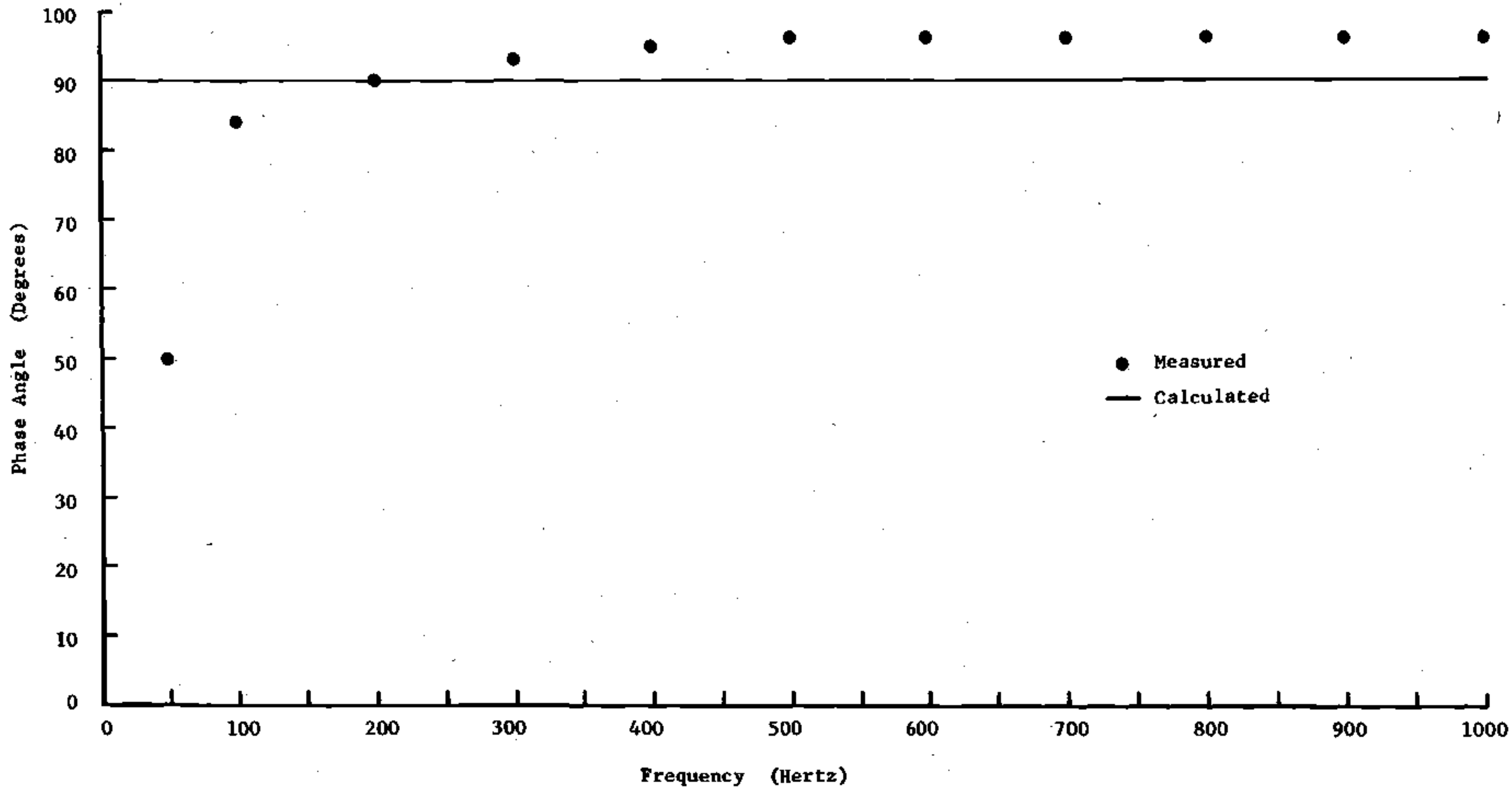


Figure 20. Comparison of Experimental Data with Desired Phase Variation for One Henry Inductor.

The short-circuit admittance matrix in Equation (260) was realized as a grounded transformerless 6-terminal 5-port passive RC network as is shown in Figure 17. Two ideal operational amplifiers were then connected to the network as shown in Figure 18, and the function $\bar{Y} = 1/s$ was realized as the admittance at port 1.

In order to obtain realistic values for the elements in Figure 17, the network was frequency and magnitude scaled. The network which was constructed had the resistor values of Figure 17 multiplied by 10^3 and the capacitor values multiplied by 10^{-6}. Figures 19 and 20 show the comparison between the predicted impedance variation at port 1 and the measured variation.

Example 4

The realization procedure of Case 5 in Chapter IV was illustrated by determining a two-port network having the prescribed open-circuit voltage transfer function

$$\frac{E_2}{E_1} = \frac{s}{s^2 + 2s + 2} \tag{261}$$

But, from Equation (130), this function was given by

$$\frac{E_2}{E_1} = \frac{-Y_{21}}{Y_{22}} \tag{262}$$

so that the two parameters Y_{21} and Y_{22} were identified in the following manner:

$$Y_{21} = \frac{-s}{s^2 + 2s + 2} \tag{263}$$

and

$$Y_{22} = 1 \tag{264}$$

The procedure of Case 5 was used to realize simultaneously the two parameters in Equation (263) and (264). Note that the proof of Case 5 makes use of the procedure in Case 3 with the proper subscript changes. Consequently, the equations referred to in the following example are understood to be those of Case 3 with the necessary subscript changes.

The parameter y_{22} was chosen to be

$$y_{22} = 2\,\frac{(s + 1)(s + 3)}{(s + 4)(s + 2)} = \frac{p_{22}}{q} \tag{265}$$

Notice that Conditions (A) through (D) which are imposed on the choice of y_{22} are not satisfied. The reason for this is that the conditions are merely sufficient and are not necessary for the realization procedure to succeed.

From Equation (171),

$$B = -(s^2 + 2s + 2)(s^2 + 2s - 2) \tag{266}$$

so that Equation (172) was used to identify

$$D_1 = s^2 + 2s + 2 \tag{267}$$

and

$$D_2 = s^2 + 2s - 2 \tag{268}$$

Equations (174), (175), and (176) gave

$$y_{23} = K_2 \frac{s^2 + 2s - 2}{s^2 + 6s + 8} = K_2 \frac{D_2}{q} \tag{269}$$

$$y_{24} = K_1 \frac{s^2 + 2s + 2}{s^2 + 6s + 8} = K_1 \frac{D_1}{q} \tag{270}$$

and

$$y_{34} = K_1 K_2 \frac{s^2 + 2s + 2}{s^2 + 6s + 8} = K_1 K_2 \frac{Q}{q} \tag{271}$$

The polynomial N_{21} was denoted by

$$N_{21} = d_1 s + d_0 \tag{272}$$

and Equation (186) was used to obtain

$$d_1 = -0.502 \tag{273}$$

and

$$d_2 = -2 \tag{274}$$

Therefore,

$$y_{21} = \frac{-0.502s - 2}{s^2 + 6s + 8} = \frac{N_{21}}{q} \tag{275}$$

Equation (180) was used to obtain

$$N_{14} = K_1\ (0.498s + 2) \tag{276}$$

so that

$$y_{14} = K_1 \frac{0.498s + 2}{s^2 + 6s + 8} \tag{277}$$

Since the four parameters y_{11}, y_{33}, y_{44}, and y_{13} were arbitrary, y_{13} was set equal to zero, and the three diagonal parameters were allowed to assume the values necessary to make rows 1, 3, and 4 of [y] satisfy the dominance condition with equality. The short-circuit admittance matrix [y] of the passive network was thus found to be

$$[y] = \begin{bmatrix} 0.3125 & -0.2500 & 0 & 0.0625 \\ -0.2500 & 0.7500 & -0.0625 & 0.0625 \\ 0 & -0.0625 & 0.0782 & 0.0156 \\ 0.0625 & 0.0625 & 0.0156 & 0.1406 \end{bmatrix} \tag{278}$$

$$+ \frac{s}{s+2}\begin{bmatrix} 0.3125 & 0.2500 & 0 & -0.0625 \\ 0.2500 & 0.5000 & 0.1250 & -0.1250 \\ 0 & 0.1250 & 0.1562 & -0.0312 \\ -0.0625 & -0.1250 & -0.0312 & 0.2185 \end{bmatrix}$$

$$+ \frac{s}{s+4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.7500 & 0.1875 & 0.3125 \\ 0 & 0.1875 & 0.2657 & 0.0782 \\ 0 & 0.3125 & 0.0782 & 0.3907 \end{bmatrix}$$

where K_1 and K_2 were both set equal to 0.25.

The above matrix was realized as the short-circuit admittance matrix of a transformerless passive RC network. Each term of the equation was realized independently as shown in Figures 21, 22, and 23, and the three networks were connected in parallel as in Figure 24, where the ideal operational amplifier is connected to ports 3 and 4 of the resulting network. The active network was then magnitude scaled by a factor of 10^4 and frequency scaled by 10^3. The network was constructed, and the data obtained from it is shown in Figures 25 and 26 along with the desired variation.

Discussion of Techniques and Errors

As was mentioned previously, the purpose of the experimental portion of this thesis was to demonstrate the validity and practicality of the realization procedures and not to develop sophisticated examples. Consequently, the construction and measurement techniques employed were

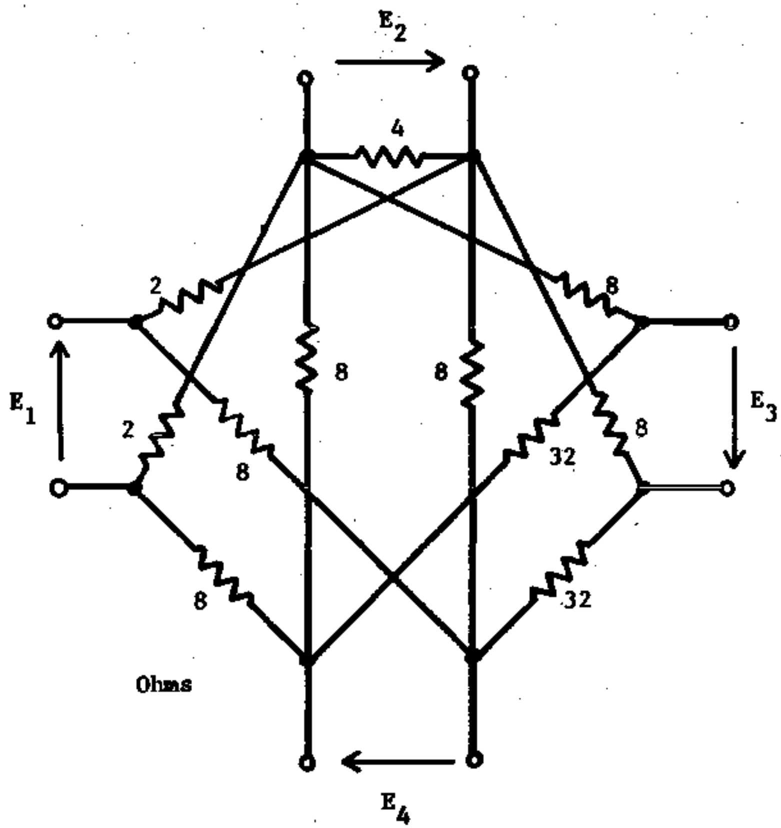


Figure 21. Network Realizing the First Term of Equation (278).

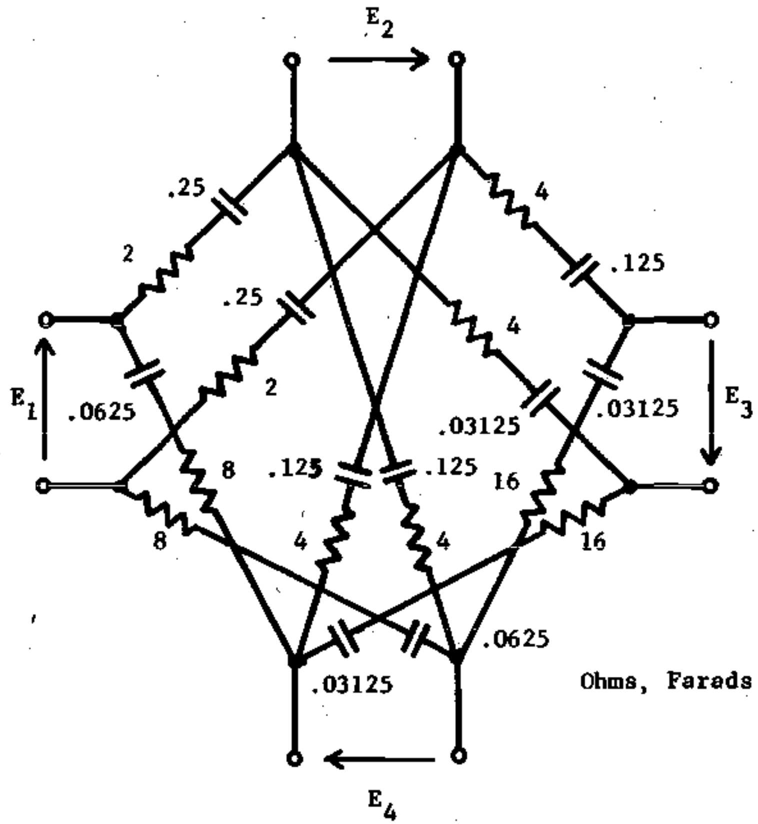


Figure 22. Network Realizing the Second Term of Equation (278).

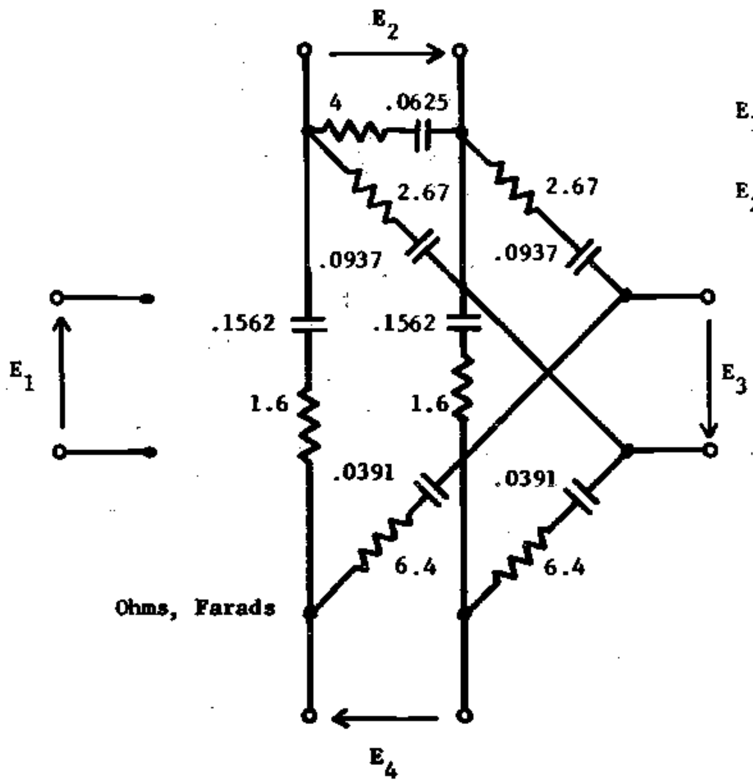


Figure 23. Network Realizing the Third Terms of Equation (278).

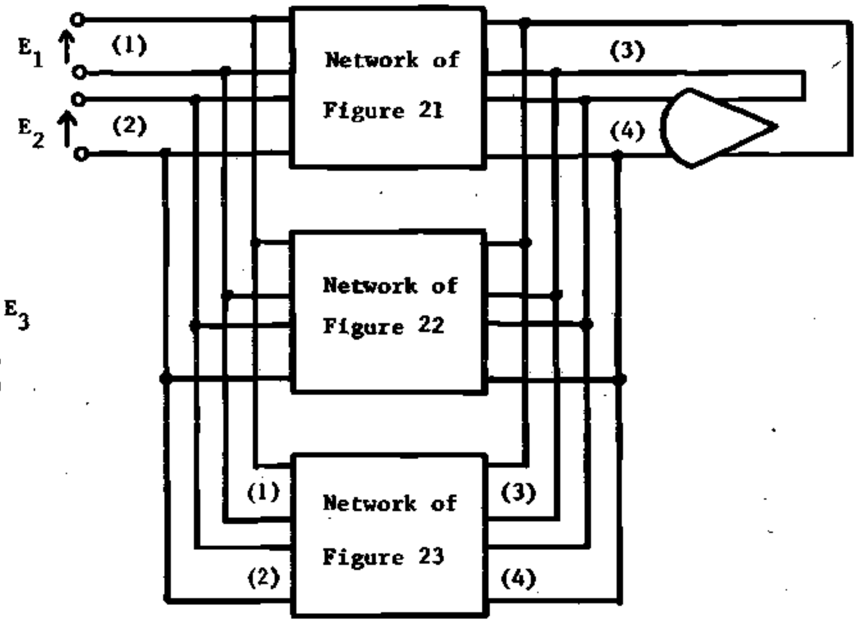


Figure 24. Network Realizing the Voltage Transfer Function of Equation (261).

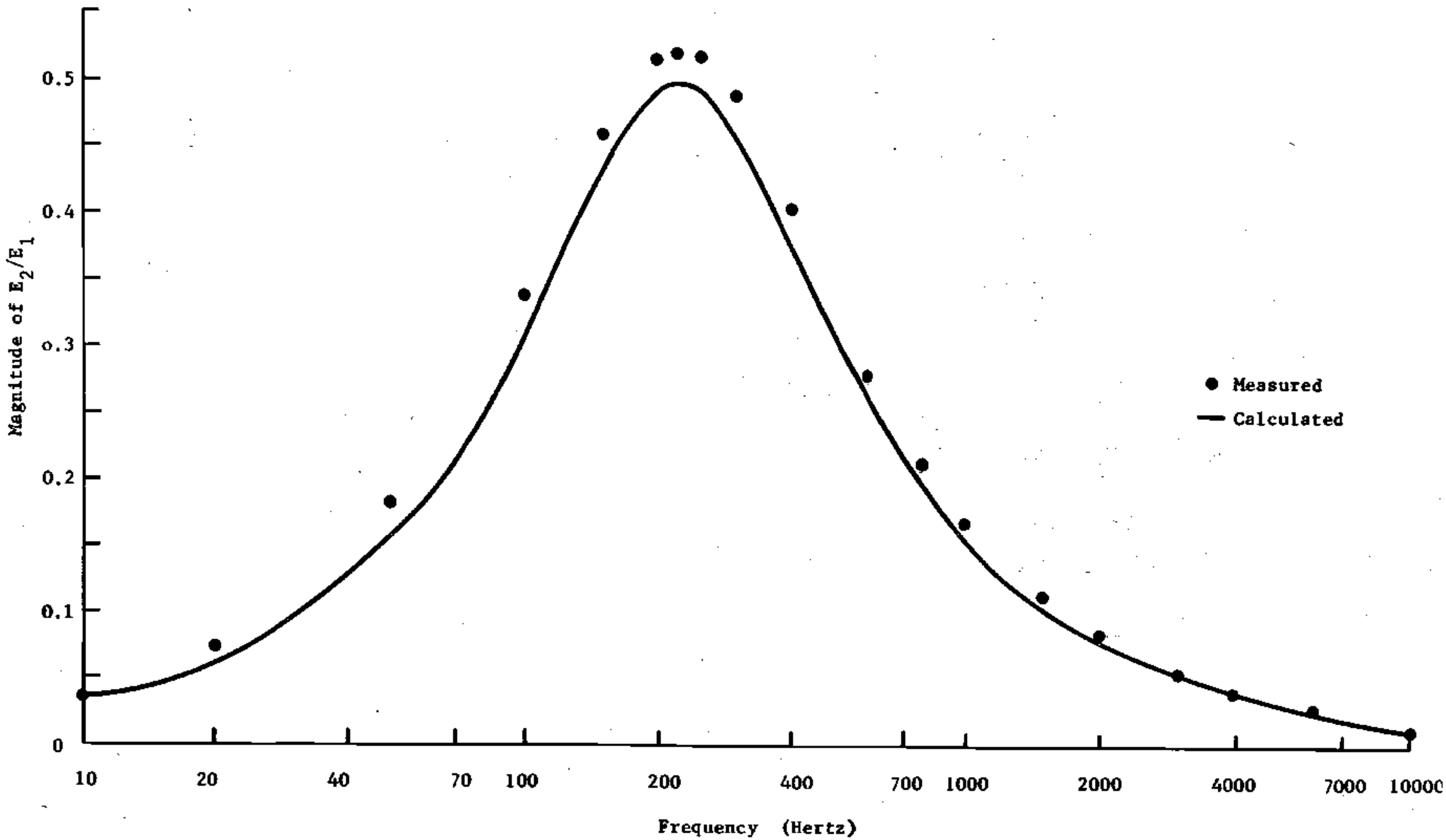


Figure 25. Comparison of Experimental Data with Desired Magnitude Variation for Network of Figure 24.

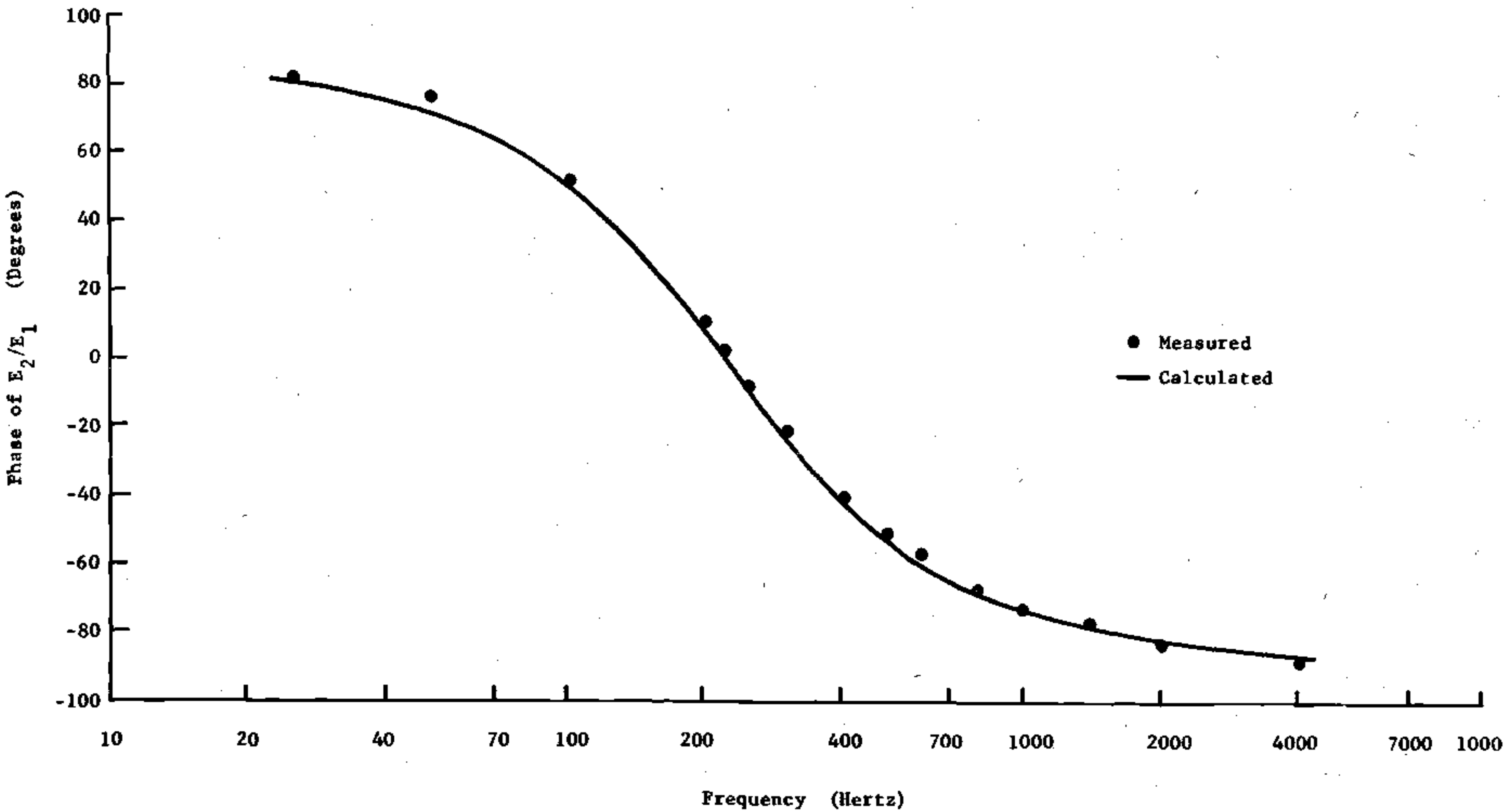


Figure 26. Comparison of Experimental Data with Desired Phase Variation for Network of Figure 24.

perhaps not the best, but the results indicate that they were sufficient to accomplish the purposes.

All of the elements which were used for construction of the RC networks deviated less than approximately 5 per cent from the designed values. One per cent precision ceramic resistors and 15 per cent tolerance mylar capacitors were employed, but all capacitor values were measured on an impedance bridge so that combinations could be chosen which were within the desired 5 per cent accuracy. Vector boards were used for construction of the networks.

The realization procedures developed in this thesis utilized ideal operational amplifiers which were assumed to possess infinite input impedance, zero output impedance, and infinite gain. In the experimental verification of the procedures, the ideal operational amplifiers were approximated with commercially available operational amplifiers. A Burr-Brown Model 1525 differential input operational amplifier was employed in all four examples and a Burr-Brown Model 1510 single-ended operational amplifier was employed as the second amplifier in Example 3. The D. C. gain of the Model 1525 was approximately 106 db, the input resistance was approximately 0.5 MΩ, and the output resistance was approximately 5 KΩ. For the Model 1510, the D. C. gain, input resistance, and output resistance were approximately 90 db, 0.5 MΩ, and 0.1 KΩ, respectively. Also, Fairchild 709 integrated operational amplifiers were substituted for the Burr-Brown amplifiers, but the data varied little even though the gains of the Fairchild amplifiers were approximately 20 db less than those of the Model 1525 amplifier.

An expression for the error due to the nonideal character of the operational amplifiers for the case $N = 1$ in procedure 2 of Chapter II was derived using a different model for the amplifier in Figure 2. The model which was used had input resistance R_1, output resistance R_2, and gain k. The short-circuit admittance matrix of the network of Figure 2 was found using this more realistic model, and an expression for the error was determined by subtracting this new short-circuit admittance matrix from the one in Equation (40). The values of R_1, R_2, and k which were given for the Burr-Brown Model 1525 amplifier and the y-parameters in Equation (260) of Example 2 were used in the error expression. The result was an involved error matrix which gave the error in each y-parameter of the network in Figure 14 as a function of the complex-frequency variable, s. It was found that the error was negligible. The error expression was again examined without restricting the gain, k, and it was noticed that as the gain was reduced by 30 to 40 db, some error was introduced. The operational amplifiers employed were compensated for stability purposes so that the gains were reduced by 6 db per octave above about 100 hertz. The conclusion was that part of the high frequency error in the measured impedances could be attributed to a reduction in the gain of the operational amplifiers. It was also noted from the error expression that the error approached zero regardless of what finite value R_1 and R_2 assumed, provided the gain approached infinity.

The magnitude of the impedances of the first three examples were measured in the following manner. A series resistor was connected to port 1 and a Hewlett Packard Model 200 CD audio oscillator was connected to the series resistor. The differential voltages across the series resistor and at the input to the original network were measured as the frequency was varied from about 10 to 1000 hertz. Because of the necessity of measuring small differential voltages, a Keithley Model 103 differential input amplifier with a voltage gain of 31.04 db was used to amplify the measured voltages and give a proportional output voltage with respect to ground. The output of the amplifier was measured with a Hewlett Packard Model 400 D vacuum tube voltmeter. Computation of the input impedance magnitude was accomplished by dividing the voltage at the original port 1 by the voltage across the series resistor and multiplying by the value of the series resistor.

Two identical Model 103 Keithley amplifiers were utilized for phase measurements. One of the amplifiers was connected to the original port 1, and the other was connected across the series resistor. The two amplifier outputs were connected to the vertical and horizontal inputs of a Hewlett Packard Model 120A oscilloscope. The phase difference between the two voltages was measured as a function of frequency using a Hewlett Packard Webb mask on the oscilloscope.

The open-circuit voltage transfer function for Example 4 was measured very simply. The oscillator was connected to port 1 of the network and a Keithley Model 103 amplifier and vacuum tube voltmeter were used to measure the input voltage and the open-circuit voltage at port 2 as functions of frequency. The input voltage was divided by the output voltage to give the magnitude of the transfer ratio. Two Keithley amplifiers, one connected to port 1 and the other to port 2, were used as before in the phase measurement.

Several difficulties were encountered while testing the network. Since the operational amplifiers saturated when the peak-to-peak voltage at their outputs exceeded approximately 25 volts, the input voltage to the network had to be restricted. Noise, particularly 60 hertz, was appreciable in the network at the outset due to the balanced networks and small differential voltages. This problem was essentially eliminated by shielding all input and power supply leads and enclosing the network in an aluminum box. Also, the oscillator voltage was maintained as large as possible without introducing saturation of the amplifiers so that the voltages in the networks would be well above the noise level.

The data obtained from Example 1, which is shown in Figures 10 and 11, agreed fairly well with the desired behavior. Deviation of the measured magnitude from its predicted value was less than 9 per cent for frequencies lower than 400 hertz. At 600 hertz the error increased to 14 per cent, and at 800 hertz it was 25 per cent. Figure 11 reveals that the phase of the impedance deviated less than 4.5 per cent from the desired 90 degrees over the frequency range 25 to 600 hertz. The error was merely 11.1 per cent at 800 hertz.

The network of Example 2 did not function as well as that of Example 1 as can be seen by an examination of the data which is displayed in Figures 15 and 16. Notice that the magnitude of the impedance deviated from its predicted value by 26.6 per cent at 500 hertz, and the error was down to 9.5 per cent at 250 hertz. The network performed well at frequencies below 200 hertz, and for this range the experimental error was probably no greater than the inaccuracy of the measurements.

The phase data shown in Figure 16 agreed much better; over the frequency range 5 to 1000 hertz, the error was less than 11.5 per cent. If the frequency range is restricted to 500 hertz as was done for the magnitude measurements, the maximum error was only 5.6 per cent.

Example 3, which is the grounded realization of the inductor, performed poorer than the two previous examples. The network was found to be very sensitive to changes in the shunt resistor across port 1. The data which is shown in Figures 19 and 20 were recorded when this shunt resistor was increased from 1713 ohms to 2150 ohms. From Figure 19 it can be seen that the error in the magnitude at 800 hertz was approximately 25 per cent and that only over the frequency range 100 to 500 hertz did the data agree well with the predicted values. The phase variation of the impedance, which is shown in Figure 20, agreed somewhat better with its desired value. The error in the phase was less than 7 per cent over the range 100 to 1000 hertz, but there was considerable error at frequencies below 100 hertz.

As can be seen in Figures 25 and 26, the network of Example 4 performed extremely well. Over the frequency range 10 to 1000 hertz, the error in the magnitude of the open-circuit voltage transfer function was less than 0.55 db, and the error in the phase was negligible. The excellent results obtained from this example indicate that perhaps there was some minor error in the impedance measurements in the other examples which did not enter into the simpler voltage transfer ratio measurement of Example 4.

Causes of the experimental error are difficult to ascertain, but most of them can probably be attributed to the following:

1. decrease of the operational amplifier gain which was compensated to fall-off at 6 db per octave,
2. deviation of elements from their designed values,
3. noise which could have seriously affected the small differential voltages,
4. errors in measurements, and
5. stray capacitances due to complicated network layout.

A thorough investigation of the sources of error was not performed since the only purposes of the experimental work were to verify the realization procedures and demonstrate their practicality. The data obtained demonstrates these two points well enough to avoid further investigation.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

The rapid development of integrated-circuit technology in the last decade has stimulated tremendous interest in the application of active RC synthesis methods to circuit design problems. Unfortunately, much of the progress which has been made in this area is of a theoretical nature and is not directly applicable to practical circuit design problems. The principal reason for the lack of applications of the new active synthesis procedures is that the active elements used are not readily available and they can not be easily approximated.

This investigation has made use of a readily available active element, the operational amplifier, which approximates reasonably well the simple model which has been employed. An interesting feature possessed by this device is that the effects of the finite input impedance and the nonzero output impedance become negligible as the gain approaches infinity. Operational amplifiers are available with gains on the order of 10^9 so that the approximation of the gain approaching infinity seems to be valid. The success of the experimental verification of the procedures indicates that they are extremely practical and should be applicable to integrated-circuit techniques.

The results of this investigation can be summarized in the following four theorems:

Theorem 1

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless active RC network containing ideal operational amplifiers (a) it is sufficient that the network contains N ideal operational amplifiers; and (b) if the matrix possesses a kth order pole of rank N off the negative-real axis, it is necessary that the network contains N ideal operational amplifiers.

Theorem 2

To realize an arbitrary N x N short-circuit admittance matrix of real rational functions in the complex-frequency variable with an N-port transformerless grounded active RC network containing ideal operational amplifiers, it is sufficient that the active network contains 2N ideal operational amplifiers.

Theorem 3

To realize two arbitrary short-circuit admittance parameters, which are real rational functions in the complex-frequency variable, with a 2-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the network contains one ideal operational amplifier.

Theorem 4

To realize N arbitrary short-circuit admittance parameters with an N-port transformerless active RC network containing ideal operational amplifiers, it is sufficient that the RC network contains one ideal operational amplifier provided

(a) the admittances are real rational functions in the complex-frequency variable and

(b) one admittance parameter is prescribed from each column (row) of the short-circuit admittance matrix of the active N-port network.

Each of these theorems has been proved, and experimental verification has been accomplished.

In the process of this investigation several problems requiring additional research have been encountered. The realization procedures in Chapter IV yield balanced networks which cannot be used for many applications. Consequently, procedures are needed for realizing simultaneously any pair of prescribed admittance functions using a transformerless grounded passive RC network and not more than three ideal operational amplifiers.

As is observed in Example 3, the sensitivity of the active RC networks to changes in the element values needs to be investigated. This is a difficult problem but one which must be undertaken if the procedures are to be used in integrated-circuit applications.

Another possible area of investigation is the feasibility of increasing the number of operational amplifiers used in the realization in order to decrease the number of capacitors and resistors or to improve the performance of the network. Because of integrated-circuit techniques, it is no longer necessary to always attempt to minimize the number of active elements.

The labor involved in the matrix realization procedures, particularly the matrix factorization step, indicates that computerization of the design method would be worthwhile. Using a computer to design the networks would enable the designer to obtain quickly several networks realizing the same prescribed function and thus give him a choice of which network to construct.

In summary, this investigation has resulted in a new approach to the active RC synthesis problem. New methods of designing practical electronic circuits have been developed and the experimental results obtained indicate that the techniques should find widespread use in integrated-circuit applications.

APPENDIX I

A METHOD OF CHOOSING $\bar{Y}_{11}$ SO THAT DET [B] HAS THE REQUIRED NUMBER OF NEGATIVE-REAL ZEROS

In order to accomplish the matrix factorization which is required in three of the realization procedures, it is sufficient that the determinant of the matrix [B], which is to be factored, contain a specific number of negative-real zeros. Sandberg[2] has shown that the determinant of [B] can always be forced to satisfy this requirement, and for the convenience of the reader, his proof will be given in this Appendix.

Let the polynomial which is to be factored be denoted by

$$[B] = q\,[P] - Q\,[p] \tag{A-1}$$

where

(A) all matrices are of size $N \times N$,

(B) $\deg p_{ii} = \deg q = M$,

(C) $\bar{Y}_{11} = [p]/q$ is the short-circuit admittance matrix of a transformerless passive RC network and it is arbitrary.

Denote $\bar{Y}_{11}$ by

$$\bar{Y}_{11} = \frac{1}{q}\begin{bmatrix} x\ p'_{11} & p_{12} & \cdots & p_{1N} \\ p_{12} & x\ p'_{22} & \cdots & p_{2N} \\ \vdots & \vdots & & \vdots \\ p_{1N} & p_{2N} & \cdots & x\ p'_{NN} \end{bmatrix} \qquad \text{(A-2)}$$

where x is a positive real constant. The polynomial det [B] can be written as

$$\det [B] = \det (q\,[P] - Q\,[p]) = (-x)^N \left(Q^N \prod_{i=1}^{N} p'_{ii} + \frac{R}{x^N}\right) \qquad \text{(A-3)}$$

where R/x^N is a polynomial with all coefficients that approach zero as x approaches infinity. Note that, as x approaches infinity, NM of the zeros of det [B] approaches the zeros of

$$\prod_{i=1}^{N} p'_{ii} \qquad \text{(A-4)}$$

The zeros of this product can be chosen to be distinct and different from those of Q. Hence, for a sufficiently large value of x, det [B] has at least NM distinct negative-real zeros that are different from those of q.

APPENDIX II

A MATRIX FACTORIZATION TECHNIQUE

In this Appendix a technique, developed by Sandberg[1,2] and later outlined by Su,[3] for factoring a matrix of polynomials will be given. The matrix which is to be factored is given by

$$[B] = [P_{ij}]\, q - [p_{ij}]\, Q = [b_{ij}] \tag{A-5}$$

where

(A) $t = \max(\deg Q, \deg [P_{ij}])$,

(B) N is the size of the two matrices $[P_{ij}]$ and $[p_{ij}]$,

(C) $M = \deg q = \deg p_{ij}$ and

(D) $[P_{ij}]$ and Q are prescribed while $[p_{ij}]$ and q may be chosen.

First, assume that $[B]$ has k simple negative-real zeros, and let them be represented by $(s + \sigma_1)$, $(s + \sigma_2)$, ..., $(s + \sigma_k)$. If it is possible to determine a nonsingular real matrix,

$$[A_i] = \begin{bmatrix} 1 & & & c_{1i} & & & \\ & \cdot & & \cdot & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & & & \\ & & & c_{ii} & & & \\ & & & \cdot & \cdot & & \\ & & & \cdot & & \cdot & \\ & & & \cdot & & & \cdot \\ & & & c_{Ni} & & & 1 \end{bmatrix} \tag{A-6}$$

such that $[B][A_i]$ has one of the factors, $(s + \sigma_i)$, of det $[B]$ contained in every polynomial element in the ith column, then

$$[B] = [B][A_i][A_i]^{-1} = [B_1] \begin{bmatrix} 1 & & & & & & & \\ & \cdot & & & & 0 & & \\ & & \cdot & & & & & \\ & & & 1 & & & & \\ & & & & (s+\sigma_i) & & & \\ & & & & & 1 & & \\ & & & & & & \cdot & \\ & & & 0 & & & & \cdot \\ & & & & & & & 1 \end{bmatrix} [A_i]^{-1} \tag{A-7}$$

Note that all elements in $[B_1]$ are identical to those of $[B]$ except in the ith column in which every polynomial is one degree lower.

Multiplying the matrix $[B]$ by $[A_i]$, evaluating the ith column at $s = -\sigma_i$, and setting all the ith column elements equal to zero, results in the following set of N equations containing N unknowns:

$$c_{1i}\, b_{11}(-\sigma_i) + c_{2i}\, b_{12}(-\sigma_i) + \cdots + c_{Ni}\, b_{1N}(-\sigma_i) = 0$$

$$c_{1i}\, b_{21}(-\sigma_i) + c_{2i}\, b_{22}(-\sigma_i) + \cdots + c_{Ni}\, b_{2N}(-\sigma_i) = 0 \tag{A-8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{1i}\, b_{N1}(-\sigma_i) + c_{2i}\, b_{N2}(-\sigma_i) + \cdots + c_{Ni}\, b_{NN}(-\sigma_i) = 0$$

This set of equations has a nontrivial solution since

$$\det\,[B(-\sigma_i)] = \det\,[b_{ij}(-\sigma_i)] = 0 \tag{A-9}$$

Thus, the c's can be determined and the matrix $[A_i]$ specified.

A necessary and sufficient condition for the existence of the inverse of $[A_i]$ is that

$$\det\,[A_i] \neq 0 \tag{A-10}$$

But, from Equation (A-6),

$$\det\,[A_i] = c_{ii} \tag{A-11}$$

so that it is merely necessary to show that $c_{ii} \neq 0$. Consider the Laplace expansion of $[b_{ij}]$ about its ith column, which yields

$$\det\,[B] = \sum_{j=1}^{N} b_{ji}\, \Delta_{ji} \tag{A-12}$$

where Δ_{ij} is the cofactor of the j,i element of [B]. Denote by d_i the greatest common polynomial factor of all the Δ_{ij} in Equation (A-12). It then follows that

$$\det [B] = d_i \sum_{j=1}^{N} b_{ji} \Delta'_{ji} \tag{A-13}$$

where $\Delta'_{ji} = \frac{\Delta_{ji}}{d_i}$. If $(s + \sigma_i)$ is a factor of $\sum_{j=1}^{N} b_{ji} \Delta'_{ji}$ and not of d_i, all (N-1)-rowed minors of [B] constructed from columns 1,2,...,(i-1), (i+1),...,N cannot vanish at $s = -\sigma_i$ since if they did, $(s+\sigma_i)$ would have been included in d_i. Thus, if c_{ii} is set equal to zero in Equation (A-8), the solutions for the c's are trivial;[16] but the solution for the complete equations are nontrivial. This implies that $c_{ii} \neq 0$, and $[A_i]$ is nonsingular.

Consequently, if det [B] has at least one zero, σ_i, which is different from those of d_i, a nonsingular matrix of constants, $[A_i]$, can be determined such that each element in the ith column of $[B][A_i]$ has a zero at $s = -\sigma_i$. Using Conditions (A) and (C) along with Equation (A-5) reveals that the maximum degree of the elements of [B] is $r = M + t$ so that the degree of the determinant of [B] is less than or equal to Nr. Since the degree of d_i can be most r(N-1), the factorization in Equation (A-7) is possible if

$$k > r(N - 1) \tag{A-14}$$

Since the determinant of the product of square matrices is equal to the product of the determinants, Equation (A-7) shows that the determinant of $[B][A_i]$ still contains all the zeros of det $[B]$. If

$$[B][A_i] = [F_{ij}] \tag{A-15}$$

then

$$\det [F_{ij}] = \sum_{j=1}^{N} F_{ju}\, \Delta_{ju} = d_u \sum_{j=1}^{N} F_{ju}\, \Delta'_{ju} \tag{A-16}$$

where $\Delta'_{ju} = \frac{\Delta_{ju}}{d_u}$ and $u \neq i$. Note that $(s + \sigma_i)$ is contained in d_u since every element in the ith column of $[F_{ij}]$ contains this factor. Thus, the requirement for this factorization to be possible on column u of $[B][A_i]$ is again the inequality in Equation (A-14). In other words, it is possible to determine a nonsingular real matrix [E] such that

$$[B][E][E]^{-1} = [B_2] \begin{bmatrix} (s+\sigma_1) & & & 0 \\ & (s+\sigma_2) & & \\ & & \ddots & \\ 0 & & & (s+\sigma_N) \end{bmatrix} [E]^{-1} \tag{A-17}$$

in which each element of $[B_2]$ is one degree lower than the corresponding element in [B] if Equation (A-14) is satisfied. All of the zeros of det [B] which are not contained in the above diagonal matrix are contained in det $[B_2]$.

If $[B_2]$ has a sufficient number of simple negative-real zeros, the factorization described above can be repeated t times. After repeating the process (t-1) times, N(t-1) of the negative-real zeros of det [B] have been used. Thus, there are k - N(t-1) negative-real zeros left. At this point the maximum degree of the matrix which is to be factored is r - (t-1) since (t-1) linear factors have been removed from [B]. The polynomial d in Equation (A-13) can therefore have no more than (N-1)[r - (t-1)] zeros. This implies that a sufficient condition for the tth repetition to be accomplished is that

$$k - N(t-1) > (N-1)[r - (t-1)] \qquad \text{(A-18)}$$

or

$$k > N(M+t) - M - 1 \qquad \text{(A-19)}$$

But, k can be made as great as NM by choosing the diagonal elements of $[p_{ij}]$ properly. Setting k = NM in Equation (A-18) gives

$$M > Nt - 1 \qquad \text{(A-20)}$$

It should be noted that the inequality in Equation (A-20) is merely a sufficient condition for the factorization and is not necessary.

If a given matrix [B] is factored, using the procedure which has been presented, so that

$$[B] = [D_1][D_2] \tag{A-21}$$

then it can be seen from Equation (A-17) that the determinant of $[D_2]$ contains only negative-real zeros which also belong to the determinant of $[B]$. Also, note that all elements of $[D_2]$ are of the same degree.

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